Mathematical Analysis of Pyramidal Flat Containers with Polygonal Base, Pyramids & Polyhedrons

(Application of HCR's Theorem and Corollary)

Mr Harish Chandra Rajpoot

Master of Technology, IIT Delhi

Introduction: In this paper, we are to derive the generalized formula to compute all the important parameters like V-cut angle (using HCR's Theorem), edge length of open end, dihedral angle (using HCR's Corollary), surface area and volume of pyramidal flat container with regular polygonal base, right pyramids and polyhedrons. We will apply these generalized formula to compute important parameters & make paper models of the pyramidal flat containers with square, regular pentagonal, hexagonal, heptagonal & octagonal bases, right pyramids and polyhedrons. The concept of making pyramidal flat containers, right pyramids and polyhedrons (bi-pyramids) is mainly based on the rotation or folding of two co-planar planes about their intersecting straight edges (as discussed in details and formulated in **HCR's Theorem**)

In brief, the procedure, of making pyramidal flat container, right pyramid or polyhedron using sheet (of desired thickness) of paper, polymer, metal or alloy which can be easily cut, bent & butt-joined at the mating lateral edges, is based on the following steps

1.) Making drawing on thin sheet of paper (we can also use sheet of polymer, metal or alloy)
2.) Cutting and removing undesired parts from sheet-drawing to get a blank
3.) Bending lateral faces (trapezoidal in case of flat container and triangular in case of pyramid) about the edges of regular polygonal base
4.) Joining the mating edges of lateral faces (gluing in case of paper, welding in case of polymer, metal or alloy)

In above procedure, the most important step is to make drawing accurately & precisely on the sheet of desired material & thickness preferably sheet of paper is most suitable for making accurate drawing. We will discuss the procedure & the drawing in details in later stages.

Pyramidal flat container with regular n-gonal base: Let's make a right pyramidal flat container of slant height l (i.e. distance between parallel sides of a lateral trapezoidal face), with a regular polygonal base with n no. of sides each of length a such that each lateral trapezoidal face is inclined at an angle θ with the plane of base.

Let's draw a circle of radius r circumscribing the regular n-gonal base $A_1A_2A_3 \dots\dots . A_{n-1}A_n$ of each side a. Draw a bigger concentric circle of radius R & extend the straight lines joining the vertices of inner regular n-gon to the centre O so that these lines intersect outer big circle at points $B_1, B_2, B_3 \dots\dots . B_{n-1}$ & B_n. After joining these points (vertices) we get another big regular n-gon $B_1B_2B_3 \dots\dots . B_{n-1}B_n$. (As shown in the fig-1)

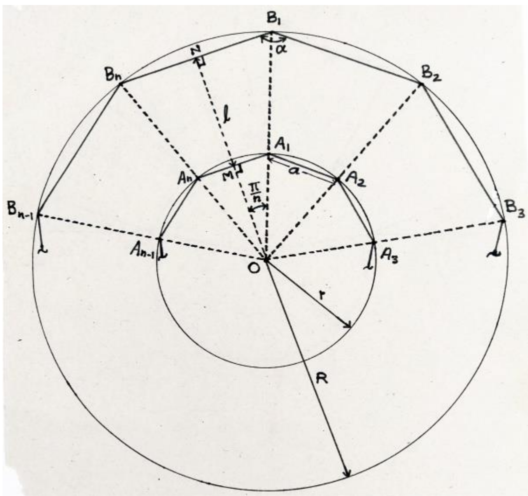


Figure-1: Inner and Outer circles of radii r & R circumscribe the inner regular n-gonal base & outer regular n-gon. MN = slant height = l

Now, interior angle (α) of regular n-gon,

$$\alpha = \frac{(n-2)\pi}{n} \quad \& \quad \angle A_1OM = \angle B_1ON = \frac{\angle A_1OA_n}{2} = \frac{\left(\frac{2\pi}{n}\right)}{2} = \frac{\pi}{n}$$

Drop a perpendicular from con-centre O to the parallel sides A_nA_1 & B_nB_1.

In right ΔOMA_1 (see fig-1 above)

$$\sin\angle A_1OM = \frac{MA_1}{OA_1} \Rightarrow \sin\frac{\pi}{n} = \frac{a/2}{r} \Rightarrow \boldsymbol{r = \frac{a}{2}cosec\frac{\pi}{n}} \quad \&$$

$$\tan\frac{\pi}{n} = \frac{\frac{a}{2}}{OM} \Rightarrow OM = \frac{a}{2}cot\frac{\pi}{n}$$

$$\therefore \boldsymbol{ON} = OM + MN = \frac{a}{2}cot\frac{\pi}{n} + l \qquad (\text{since,} \quad \text{MN} = \text{slant height} = l\ )$$

Similarly, in right ΔONB_1 (see fig-1 above)

$$\cos\angle B_1ON = \frac{ON}{OB_1} \quad \Rightarrow \cos\frac{\pi}{n} = \frac{\left(\frac{a}{2}cot\frac{\pi}{n} + l\right)}{R}$$

$$\boldsymbol{R = \frac{a}{2}cosec\frac{\pi}{n} + l\,sec\frac{\pi}{n}} \quad (\text{value of R gives the size of circular sheet to make desired container})$$

Since each trapezoidal lateral face of pyramidal flat container is inclined at angle θ with the plane of regular n-gonal base hence we need to cut V-part (included angle δ) symmetrically from each vertex of inner polygon such that δ is bisected by radial dotted lines (as shown by shaded area in the figure-2).

Derivation of V-cut angle (δ): V-cut angle δ, required to cut remove V-parts and rotate two co-planar planes (i.e. trapezoidal faces) $A_1B_1B_nA_n$ & $A_1B_1B_2A_2$ (initially intersecting at common edge A_1B_1) through an angle θ about their straight edges A_nA_1 & A_1A_2 respectively intersecting each other at an angle α so that their new edges A_1P & A_1Q respectively coincide at $A_1A'_1$, is given by **HCR's Theorem** as follows

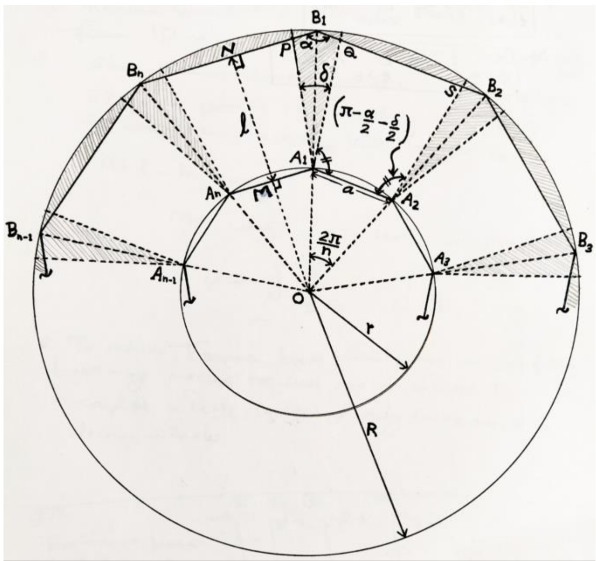


Figure-2: δ is V-cut angle of V-parts which is bisected by radial dotted lines joining the vertices of regular n-gons to the centre O. The shaded area is to be cut removed from big circle of radius R. QS is the edge length of regular n-gonal open end of pyramidal flat container

$$\boldsymbol{\delta = 2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right) - \alpha}$$

Setting the value, $\alpha = \frac{(n-2)\pi}{n}$, we get

$$\delta = 2\tan^{-1}\left(\sec\theta\tan\frac{(n-2)\pi}{2n}\right) - \frac{(n-2)\pi}{n}$$

$$= 2\tan^{-1}\left(\sec\theta\tan\left(\frac{\pi}{2} - \frac{\pi}{n}\right)\right) - \pi + \frac{2\pi}{n}$$

$$= 2\tan^{-1}\left(\sec\theta\cot\frac{\pi}{n}\right) - \pi + \frac{2\pi}{n}$$

$$= 2\cot^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right) - \pi + \frac{2\pi}{n} \qquad \left(\text{since,}\quad \tan^{-1}x = \cot^{-1}\frac{1}{x}\right)$$

$$= 2\left(\frac{\pi}{2} - \tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right)\right) - \pi + \frac{2\pi}{n} \qquad \left(\text{since,}\quad \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x\right)$$

$$= \pi - 2\tan^{-1}\left(\cos\theta \tan\frac{\pi}{n}\right) - \pi + \frac{2\pi}{n}$$

$$= \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta \tan\frac{\pi}{n}\right)$$

Hence, **V-cut angle δ, required to cut remove V-parts & rotate each two co-planar planes about their intersecting straight edges i.e. edges of regular n-gonal base of pyramidal flat container**, is given as

$$\boldsymbol{\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta \tan\frac{\pi}{n}\right)}$$

Derivation of edge length (a_1) of regular n-gonal open end of pyramidal flat container: QS is the edge length of regular n-gonal open end (see above fig-2)

From figure-2 above,

$$A_1B_1 = OB_1 - OA_1 = R - r$$

$$A_1B_1 = \frac{a}{2} cosec\frac{\pi}{n} + l\sec\frac{\pi}{n} - \frac{a}{2} cosec\frac{\pi}{n}$$

$$A_1B_1 = l\sec\frac{\pi}{n}$$

In ΔA_1PB_1 (See fig-2 above),

$$\angle \boldsymbol{PA_1B_1} = \frac{\angle PA_1Q}{2} = \frac{\delta}{2} \quad \& \quad \angle PB_1A_1 = \frac{\angle PB_1Q}{2} = \frac{\alpha}{2} \qquad \forall\ \alpha = \frac{(n-2)\pi}{n}$$

$$\therefore\ \angle \boldsymbol{A_1PB_1} = \pi - \angle PB_1A_1 - \angle PA_1B_1$$

$$= \pi - \frac{\alpha}{2} - \frac{\delta}{2}$$

Now, using Sine rule in ΔA_1PB_1 (See fig-2 above)

$$\frac{PB_1}{\sin\angle PA_1B_1} = \frac{A_1B_1}{\sin\angle A_1PB_1}$$

$$\frac{PB_1}{\sin\frac{\delta}{2}} = \frac{l\sec\frac{\pi}{n}}{\sin\left(\pi - \frac{\alpha}{2} - \frac{\delta}{2}\right)}$$

$$PB_1 = \frac{l\sec\frac{\pi}{n}\sin\frac{\delta}{2}}{\sin\left(\frac{\alpha+\delta}{2}\right)}$$

$$= \frac{l\sec\frac{\pi}{n}\sin\left(\frac{\alpha+\delta}{2} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha+\delta}{2}\right)}$$

$$= \frac{l\sec\frac{\pi}{n}\left(\sin\left(\frac{\alpha+\delta}{2}\right)\cos\frac{\alpha}{2} - \cos\left(\frac{\alpha+\delta}{2}\right)\sin\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha+\delta}{2}\right)}$$

$$= l\,sec\frac{\pi}{n}\left(\cos\frac{\alpha}{2} - \frac{\cos\left(\frac{\alpha+\delta}{2}\right)\sin\frac{\alpha}{2}}{\sin\left(\frac{\alpha+\delta}{2}\right)}\right)$$

$$PB_1 = l\,sec\frac{\pi}{n}\left(\cos\frac{\alpha}{2} - \frac{\sin\frac{\alpha}{2}}{\tan\left(\frac{\alpha+\delta}{2}\right)}\right) \quad \dots\dots\dots\dots\dots(1)$$

Now, from HCR's Theorem, we know that V-cut angle is given as

$$\delta = 2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right) - \alpha$$

$$\alpha + \delta = 2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right)$$

$$\frac{\alpha+\delta}{2} = \tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right)$$

$$\tan\left(\frac{\alpha+\delta}{2}\right) = \sec\theta\tan\frac{\alpha}{2}$$

Setting above value of $\tan\left(\frac{\alpha+\delta}{2}\right)$ in eq(1), we get

$$PB_1 = l\,sec\frac{\pi}{n}\left(\cos\frac{\alpha}{2} - \frac{\sin\frac{\alpha}{2}}{\sec\theta\tan\frac{\alpha}{2}}\right)$$

$$= l\,sec\frac{\pi}{n}\left(\cos\frac{\alpha}{2} - \frac{\sin\frac{\alpha}{2}\cos\theta\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\right)$$

$$= l\,sec\frac{\pi}{n}\left(\cos\frac{\alpha}{2} - \cos\theta\cos\frac{\alpha}{2}\right)$$

$$= l\,sec\frac{\pi}{n}\cos\frac{\alpha}{2}(1-\cos\theta)$$

$$= l\,sec\frac{\pi}{n}\cos\frac{(n-1)\pi}{2n}(1-\cos\theta)$$

$$= \frac{l}{\cos\frac{\pi}{n}}\cos\left(\frac{\pi}{2}-\frac{\pi}{n}\right)(1-\cos\theta)$$

$$= \frac{l}{\cos\frac{\pi}{n}}\sin\frac{\pi}{n}(1-\cos\theta)$$

$$= l\,\tan\frac{\pi}{n}(1-\cos\theta)$$

From symmetry in fig-2 above, we have

$$SB_2 = B_1Q = PB_1 = l\,\tan\frac{\pi}{n}(1-\cos\theta)$$

In right ΔOEB_1 (See figure-3 below)

$$\sin\angle CA_1B_1 = \frac{B_1E}{OB_1} \quad \Rightarrow \sin\frac{\pi}{n} = \frac{B_1E}{R} \quad \Rightarrow B_1E = R\sin\frac{\pi}{n}$$

$$B_1E = \left(\frac{a}{2}cosec\frac{\pi}{n} + l\ sec\frac{\pi}{n}\right)\sin\frac{\pi}{n} \qquad \text{(setting value of R)}$$

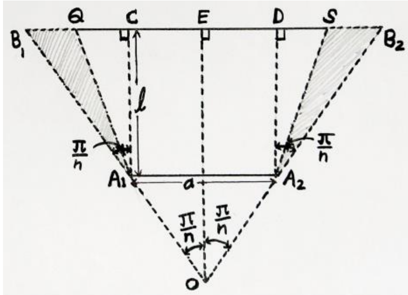


Figure-3: This is a part of figure-2 above. A_1QSA_2 is trapezoidal face of pyramidal flat container after cut-removing undesired area

$$B_1E = \frac{a}{2} + l\tan\frac{\pi}{n}$$

$$\therefore\ B_1B_2 = 2B_1E = 2\left(\frac{a}{2} + l\tan\frac{\pi}{n}\right) = a + 2l\tan\frac{\pi}{n}$$

$$\therefore QS = B_1B_2 - B_1Q - SB_2$$

$$= B_1B_2 - 2B_1Q \qquad \text{(since, } SB_2 = B_1Q\text{)}$$

$$= a + 2l\tan\frac{\pi}{n} - 2\left(l\ \tan\frac{\pi}{n}(1-\cos\theta)\right)$$

$$= a + 2l\tan\frac{\pi}{n} - 2l\tan\frac{\pi}{n} + 2l\cos\theta\ \tan\frac{\pi}{n}$$

$$= a + 2l\cos\theta\ \tan\frac{\pi}{n}$$

Hence, the **edge length (a_1) of regular n-gonal open end of pyramidal flat container**, is given as

$$\boldsymbol{a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n}}$$

Now, make the V-cut angle δ at each vertex of small regular n-gon such that δ is bisected by radial dotted line (as shown in fig-2 above) & mark V-parts and undesired area of big circle with radius R (See fig-2 above)

After cut-removing undesired area from big circle we get a blank of paper (as shown in the figure-4). Then, each of trapezoidal faces is rotated about the edges of regular n-gonal base $A_1A_2A_3 \ldots \ldots A_{n-1}A_n$ until their new edges (generated after cutting V-parts) coincide with one another. Then butt-join the mating lateral edges using suitable adhesive to get a desired pyramidal flat container (as shown in fig-5 below).

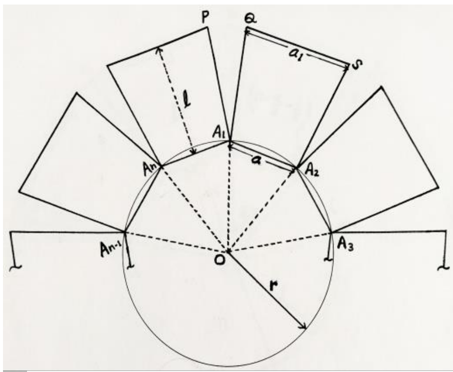


Figure-4: Blank of paper sheet after cut-removing undesired area from big circle

Derivation of dihedral angle (θ_d) between trapezoidal lateral faces of pyramidal flat container: The dihedral angle θ_d, between two cut planes (as shown in fig-4) rotated through an angle θ about their straight edges intersecting each other at an angle α so that their new edges coincide, is given by **HCR's Corollary** as follows

$$\boldsymbol{\cos\frac{\theta_d}{2} = \sin\theta\cos\frac{\alpha}{2}}$$

Now, setting the value $\alpha = \frac{(n-2)\pi}{n}$ in above formula, we get

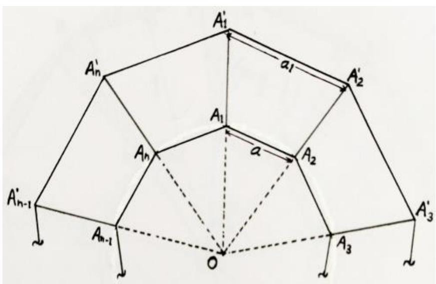


Figure-5: Top view of flat container after rotating lateral faces until their new edges coincide. Ex. new edges PA_1 & QA_1 coincide at A'_1A_1 (i.e. lateral edge)

$$\cos\frac{\theta_d}{2}=\sin\theta\cos\frac{(n-2)\pi}{2n}$$

$$\cos\frac{\theta_d}{2}=\sin\theta\cos\left(\frac{\pi}{2}-\frac{\pi}{n}\right)$$

$$\cos\frac{\theta_d}{2}=\sin\theta\sin\frac{\pi}{n}$$

$$\theta_d=2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right)$$

Hence, the **dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces of pyramidal flat container,** is given as

$$\boldsymbol{\theta_d=2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right)}$$

Special case: If the trapezoidal lateral faces are rotated through right angle i.e. $\theta=90^o$ then the dihedral angle θ_d between the consecutive lateral faces of pyramidal flat container is given as follows

$$\theta_d=2\cos^{-1}\left(\sin 90^o\sin\frac{\pi}{n}\right)=2\cos^{-1}\left(\sin\frac{\pi}{n}\right)=2\cos^{-1}\left(\cos\left(\frac{\pi}{2}-\frac{\pi}{n}\right)\right)=2\left(\frac{\pi}{2}-\frac{\pi}{n}\right)=\frac{(n-2)\pi}{n}$$

Derivation of length (L) of each lateral edge of pyramidal flat container: The trapezoidal lateral face A_1QSA_2 with parallel sides $a\,\&\,a_1$ at a normal distance l is shown in fig-6. Drop perpendiculars $A_1C\ \&\ A_2D$ from vertices $A_1\ \&\ A_2$ to the QS to get two congruent right triangles by symmetry. Let $A_1Q=A_2S=\text{edge length}=\text{L}$. Now, we have (See fig-6)

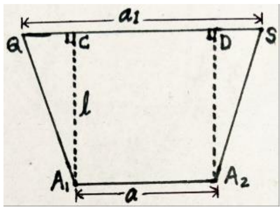


Figure-6: Trapezoidal face of pyramidal flat container. $\boldsymbol{A_1Q=A_2S=\text{edge length}=L}$

$$QC=DS=\frac{a_1-a}{2}$$

Using Pythagorean Theorem in right ΔA_1CQ as follows

$$(A_1Q)^2=(QC)^2+(A_1C)^2$$

$$A_1Q=\sqrt{(QC)^2+(A_1C)^2}$$

$$=\sqrt{\left(\frac{a_1-a}{2}\right)^2+(l)^2}$$

$$=\sqrt{\left(\frac{a+2l\cos\theta\ \tan\frac{\pi}{n}-a}{2}\right)^2+l^2}\qquad\qquad(\text{setting the value of }a_1)$$

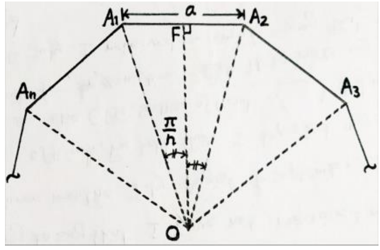
$$=\sqrt{l^2\cos^2\theta\tan^2\frac{\pi}{n}+l^2}$$

$$=l\sqrt{1+\cos^2\theta\tan^2\frac{\pi}{n}}$$

Hence, the **length (L) of each lateral edge of pyramidal flat container,** is given as

$$\boldsymbol{L=l\sqrt{1+\cos^2\theta\tan^2\frac{\pi}{n}}}$$

Derivation of surface area (A_s) of pyramidal flat container: We know that pyramidal flat container has a regular n-gonal base $A_1A_2A_3 \dots \dots . A_{n-1}A_n$ with each side a (see regular n-gonal base in fig-7) and each of its lateral faces as a trapezoid with parallel sides a & a_1 at a normal distance l (See trapezoidal face in fig-8 below).



Fogure-7: Regular n-gonal base is divided into n no. of congruent isosceles triangles

Drop a perpendicular OF from centre O to the side A_1A_2 which bisects angle $\frac{2\pi}{n}$

In right ΔOFA_1 (see fig-7 above)

$$\cot\frac{\pi}{n} = \frac{a/2}{OF} \ \Rightarrow \ OF = \frac{a}{2}\cot\frac{\pi}{n}$$

The area of isosceles ΔOA_1A_2 with base $A_1A_2 = a$ & normal height $OF = \frac{a}{2}\cot\frac{\pi}{n}$

$$= \frac{1}{2}(a)\left(\frac{a}{2}\cot\frac{\pi}{n}\right) = \frac{a^2}{4}\cot\frac{\pi}{n}$$

Since regular n-gonal base is divided into n no. of congruent isosceles triangle hence **area of regular n-gonal base of pyramidal flat container**

$$= n(\text{area of isosceles } \Delta OA_1A_2) = n\left(\frac{a^2}{4}\cot\frac{\pi}{n}\right) = \boldsymbol{\frac{1}{4}n\,a^2\cot\frac{\pi}{n}}$$

Now, the area of trapezoidal lateral face A_1QSA_2 with parallel sides a & a_1 at a normal distance l (see fig-8)

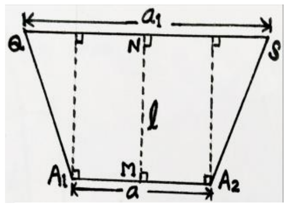


Figure-8: Trapezoidal face of pyramidal flat container

$$= \frac{1}{2}(\text{sum of parallel sides})(\text{normal distance})$$

$$= \frac{1}{2}(a + a_1)(l)$$

$$= \frac{1}{2}\left(a + a + 2l\cos\theta\ \tan\frac{\pi}{n}\right)l$$

$$= l\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right)$$

Since the lateral surface of pyramidal flat container consists of n no. of congruent trapezoidal faces, hence **total area of lateral surface of pyramidal flat container**

$$= n(\text{Area of trapezium } A_1QSA_2\ )$$

$$= \boldsymbol{nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right)}$$

Now, total surface area of regular n-gonal base and lateral surface

$$= \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right)$$

Hence, the **total surface area (A_s) of pyramidal flat container (base +lateral surface)**, is given as

$$\boldsymbol{A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\,\tan\frac{\pi}{n}\right)}$$

Derivation of volume (V) of pyramidal flat container with regular n-gonal base: The pyramidal flat container is actually a frustum of hollow right pyramid with regular n-gonal base. If we extend the lateral edges of pyramidal flat container, they intersect one another at point P (apex) & thus we get an imaginary hollow right pyramid with regular n-gonal base of each side a_1 & normal height H (as shown in the fig-9). Let $O_1O = h$ be the normal height of pyramidal flat container.

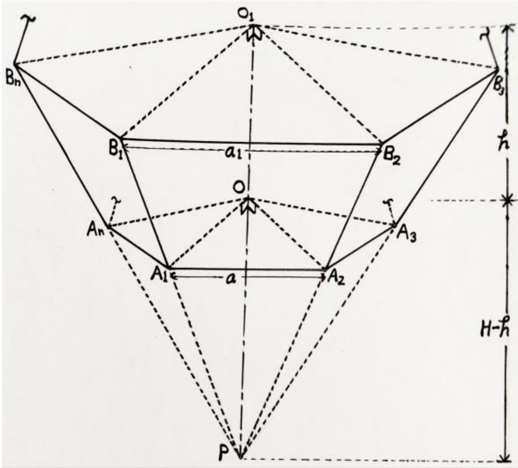


Figure-9: Lateral edges of pyramidal flat container are extended to intersect at point P to get an imaginary hollow right pyramid of height *H*

Drop a perpendicular from apex P to the planes of regular n-gonal ends of container which passes through the centres O & O_1 . Join the centres O & O_1 to the vertices of respective regular n-gons by dotted straight lines (As shown in the fig-9)

In similar right triangles ΔPO_1B_1 & ΔPOA_1 (See fig-9)

$$\frac{PB_1}{PA_1} = \frac{O_1P}{OP} \qquad \ldots\ldots\ldots..(I)$$

In similar isosceles triangles ΔPB_1B_2 & ΔPA_1A_2 (See fig-9)

$$\frac{PB_1}{PA_1} = \frac{B_1B_2}{A_1A_2} \qquad \ldots\ldots\ldots..(II)$$

Equating (I) & (II), we get

$$\frac{O_1P}{OP} = \frac{B_1B_2}{A_1A_2}$$

$$\frac{H}{H-h} = \frac{a_1}{a}$$

$$aH = a_1H - a_1h$$

$$H = \frac{a_1h}{a_1 - a}$$

$$H = \frac{\left(a + 2l\cos\theta\,\tan\frac{\pi}{n}\right)h}{a + 2l\cos\theta\,\tan\frac{\pi}{n} - a} \qquad \text{(setting the value of } a_1\text{)}$$

$$H = \frac{h\left(a + 2l\cos\theta\,\tan\frac{\pi}{n}\right)}{2l\cos\theta\,\tan\frac{\pi}{n}}$$

$$\boldsymbol{H = h\left(1 + \frac{a}{2l}\sec\theta\,\cot\frac{\pi}{n}\right)}$$

Above is the general formula to compute the height of imaginary hollow right pyramid complementary of pyramidal flat container.

The volume (V) of pyramidal flat container with regular n-gonal base will be equal to the difference of volumes of big hollow right pyramid of normal height H & small hollow right pyramid of normal height $(H-h)$ (see above fig-9), is given as

$$V=\frac{1}{3}(\text{Area of regular polygon with side } a_1)(H)-\frac{1}{3}(\text{Area of regular polygon with side a})(H-h)$$

$$=\frac{1}{3}\left(\frac{1}{4}n{a_1}^2\cot\frac{\pi}{n}\right)H-\frac{1}{3}\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)(H-h)\qquad(\text{area of regular polygon derived above in fig. 7})$$

$$=\frac{n}{12}\cot\frac{\pi}{n}({a_1}^2H-a^2H+a^2h)$$

$$=\frac{n}{12}\cot\frac{\pi}{n}\left(({a_1}^2-a^2)H+a^2h\right)$$

$$=\frac{n}{12}\cot\frac{\pi}{n}\left(\left(\left(a+2l\cos\theta\,\tan\frac{\pi}{n}\right)^2-a^2\right)h\left(1+\frac{a}{2l}\sec\theta\,\cot\frac{\pi}{n}\right)+a^2h\right)\qquad(\text{setting values of } a_1 \,\&\, H)$$

$$=\frac{nh}{12}\cot\frac{\pi}{n}\left(\left(a^2+4l^2\cos^2\theta\tan^2\frac{\pi}{n}+4al\cos\theta\,\tan\frac{\pi}{n}-a^2\right)\left(1+\frac{a}{2l}\sec\theta\,\cot\frac{\pi}{n}\right)+a^2\right)$$

$$=\frac{nh}{12}\cot\frac{\pi}{n}\left(\left(4l^2\cos^2\theta\tan^2\frac{\pi}{n}+4al\cos\theta\,\tan\frac{\pi}{n}\right)\left(1+\frac{a}{2l}\sec\theta\,\cot\frac{\pi}{n}\right)+a^2\right)$$

$$=\frac{nh}{12}\cot\frac{\pi}{n}\left(4l^2\cos^2\theta\tan^2\frac{\pi}{n}+4al\cos\theta\,\tan\frac{\pi}{n}+2al\cos\theta\,\tan\frac{\pi}{n}+2a^2+a^2\right)$$

$$=\frac{nh}{12}\cot\frac{\pi}{n}\left(3a^2+4l^2\cos^2\theta\tan^2\frac{\pi}{n}+6al\cos\theta\,\tan\frac{\pi}{n}\right)$$

$$=\frac{nh}{12}\left(3a^2\cot\frac{\pi}{n}+4l^2\cos^2\theta\tan\frac{\pi}{n}+6al\cos\theta\right)$$

The normal height h of pyramidal flat container can be expressed in terms of slant height l & the angle of inclination θ of each trapezoidal face with the plane of base (see fig-10)

In right ΔNTM,

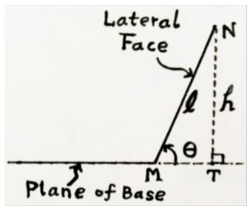


Figure-10: Each lateral trapezoidal face is inclined with the plane of base at an angle θ

$$\sin\theta={}^{h}/_{l}\quad\Rightarrow\quad h=l\sin\theta$$

Setting the value of normal height h of container in terms of slant height l in expression of volume, we get

$$V=\frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n}+4l^2\cos^2\theta\tan\frac{\pi}{n}+6al\cos\theta\right)$$

Hence, the **volume (V) of pyramidal flat container with regular n-gonal base of each side a, slant height l & angle of inclination θ of each trapezoidal face with the plane of base,** is given as

$$\boldsymbol{V=\frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n}+4l^2\cos^2\theta\tan\frac{\pi}{n}+6al\cos\theta\right)}$$

Special case: If the angle of inclination of each trapezoidal face with the plane of base becomes 90^o i.e. $\theta = 90^o$, then the volume of pyramidal flat container is obtained by setting $\theta = 90^o$ in above formula as follows

$$V = \frac{nl\sin 90^o}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2 90^o\tan\frac{\pi}{n} + 6al\cos 90^o\right)$$

$$= \frac{nl}{12}\left(3a^2\cot\frac{\pi}{n}\right)$$

$$= \left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)l$$

$$= \left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)h \qquad\qquad (\text{since,}\quad h = l\sin 90^o = l)$$

$$= (\text{Area of polygon with } n \text{ no. of sides each of length } a\,)(\text{Normal height})$$

$$= \text{Volume of right prism with regular polygonal cross section of side } a \text{ \& height } h$$

The above result is true because a pyramidal flat container with lateral faces inclined at right angle with the plane of regular n-gonal base of side a becomes a right prism with regular n-gonal cross-section of side a & height h. This result verifies the above formula of volume of pyramidal flat container.

General steps, for making a pyramidal flat container having regular n-gonal base of each side a, slant height l & angle of inclination θ of each trapezoidal face with the plane of base, using a thin sheet of paper, polymer, metal or alloy of desired thickness which can easily be cut, bent and butt-joined at mating edges, are as follows

Step 1: Draw a circle with radius $r = \frac{a}{2}cosec\frac{\pi}{n}$ & centre 'O' on the sheet of desired material and thickness.

Step 2: Draw another circle of radius $R = \frac{a}{2}cosec\frac{\pi}{n} + l\sec\frac{\pi}{n}$ concentric with small circle of radius r on the sheet

Step 3: Take aperture angle $\frac{2\pi}{n}$ & draw a regular polygon $A_1A_2A_3\ldots\ldots A_{n-1}A_n$ with n no. of sides each of length a circumscribed by the small circle of radius r (as shown in fig-1 above)

Step 4: Join the vertices $A_1, A_2, A_3, \ldots\ldots A_{n-1}$ & A_n to the centre O by dotted straight lines which when extended intersect big circle at the points $B_1, B_2, B_3 \ldots\ldots B_{n-1}$ & B_n. Joint these points by straight lines to get a big regular n-gon $B_1B_2B_3\ldots\ldots B_{n-1}B_n$ (as shown in fig-1 above)

Step 5: Make V-cut angle $\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right)$ at each vertex of small regular n-gon such that δ is bisected by the radial line passing through that vertex (as shown in fig-2 above)

Step 6: Mark V-cut parts & undesired area (as shaded in the fig-2 above) & cut remove undesired area from big circular sheet of radius R to get a blank of sheet (as shown in the fig-4 above)

Step 7: Rotate or bend each of trapezoidal faces about the edges of regular n-gonal base $A_1A_2A_3\ldots\ldots A_{n-1}A_n$ until their new edges (generated after cutting V-parts) coincide with one another. Then butt-join the mating lateral edges using a suitable adhesive or welding process to get a desired pyramidal flat container (top view of flat container is shown in fig-5 above)

Thus, a desired pyramidal flat container (as obtained above), with regular n-gonal base of each side a, slant height l & angle of inclination θ of each trapezoidal face with the plane of base, has the following important parameters

1. Edge length (a_1) of regular n-gonal open end of pyramidal flat container, is given as

$$a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n}$$

2. Length (L) of each lateral edge of pyramidal flat container, is given as

$$L = l\sqrt{1+\cos^2\theta\tan^2\frac{\pi}{n}}$$

3. Normal height (h) of pyramidal flat container, is given as

$$h = l\sin\theta$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces of pyramidal flat container, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right)$$

5. Total surface area (A_s) of pyramidal flat container (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right)$$

6. Volume (V) of pyramidal flat container, is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

We will now make some typical paper-models of pyramidal flat container using general steps as mentioned above & the generalized formula as derived above.

1. Pyramidal flat container with square base: Let's make a paper model of pyramidal flat container with square base of each side 5.4 cm , slant height 4.5 cm and each trapezoidal lateral face is inclined with the plane of base at an angle 65^o.

Given data: $n = \text{number of sides of square base} = 4$

$a = \text{edge length of square base} = 5.4\text{ cm}$

$l = \text{slant height of pyramidal flat container} = 4.5\text{ cm}$

$\theta = \text{angle of inclination of each trapezoidal lateral face with the plane of base} = 65^o$

Following steps (as shown in pictures below) are used to make a pyramidal flat container (of desired dimensions) with square base using sheet of paper

1.) Calculate radius (r) of small circle

$$r = \frac{a}{2}cosec\frac{\pi}{n} = \frac{5.4}{2}cosec\frac{\pi}{4} = 3.82\ cm$$

Draw a circle with radius $r = 3.82\ cm$ on the sheet of paper (See fig-11)

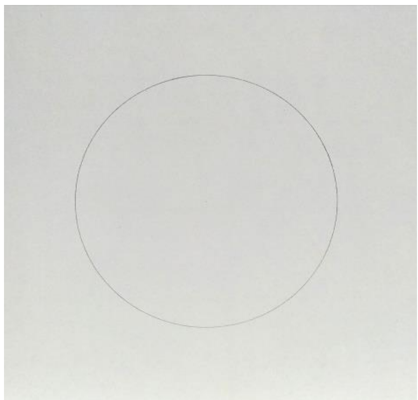


Figure-11: A circle of radius 3.82 cm is drawn on the sheet of paper

2.) Calculate radius (R) of big circle

$$R = \frac{a}{2} cosec \frac{\pi}{n} + l \sec \frac{\pi}{n} = \frac{5.4}{2} cosec \frac{\pi}{4} + 4.5 \sec \frac{\pi}{4} = 10.18\ cm$$

Draw another circle of radius $R = 10.18\ cm$ concentric with small circle (See fig-12)

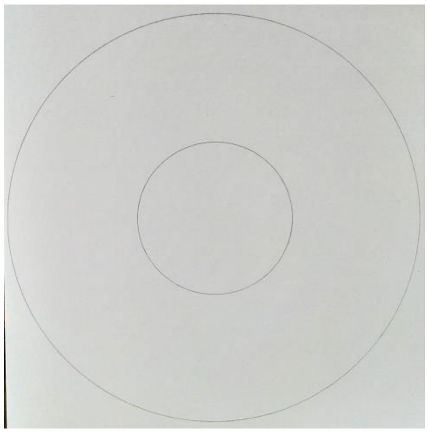


Figure-12: A big circle of radius 10.18 cm drawn on the sheet of paper is concentric with small circle of radius 3.82 cm

3.) Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{4} = 90^o$ & draw a square of each side $a = 5.4\ cm$ circumscribed by the small circle of radius $r = 3.82\ cm$ (as shown in fig-13). Join the vertices of square to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 10.18 cm (As shown in fig-14 below)

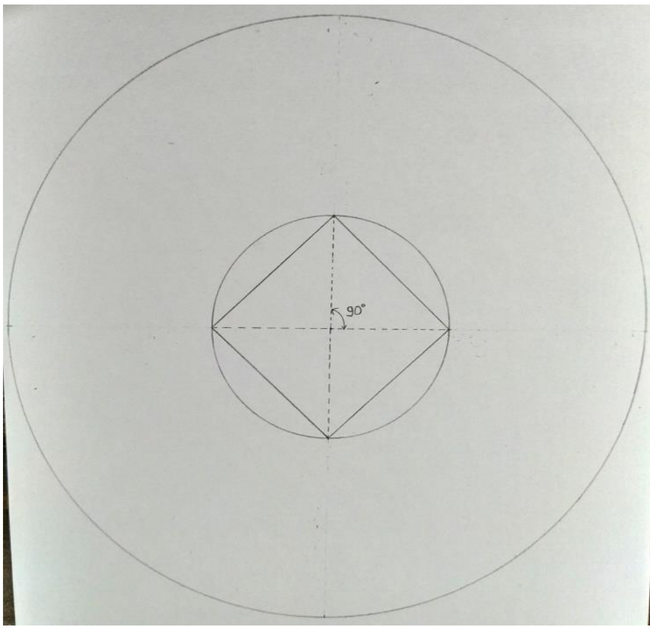


Figure-13: A square of side 5.4 cm drawn on the sheet of paper is circumscribed by small circle of radius 3.82 cm

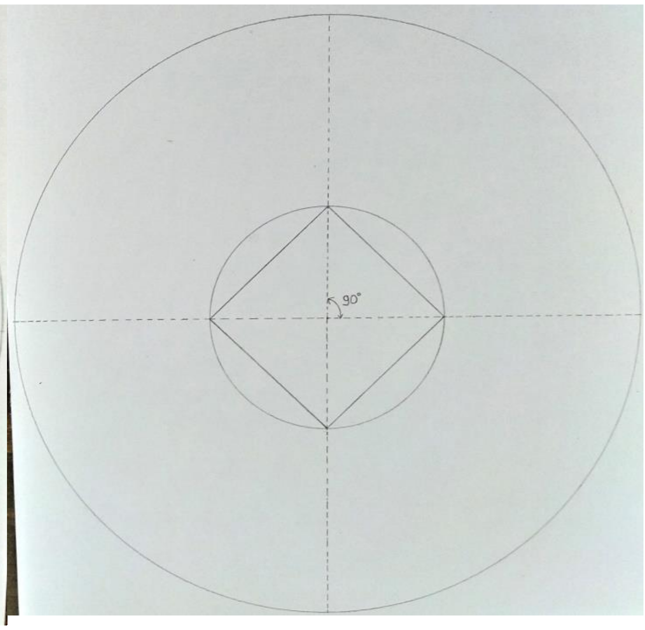


Figure-14: Dotted straight lines joining the vertices of square to the centre are extended so as to intersect big circle of radius 10.18 cm

4.) Join the points of intersection of radial lines & big circle by the straight lines to get a big square (See fig-15)

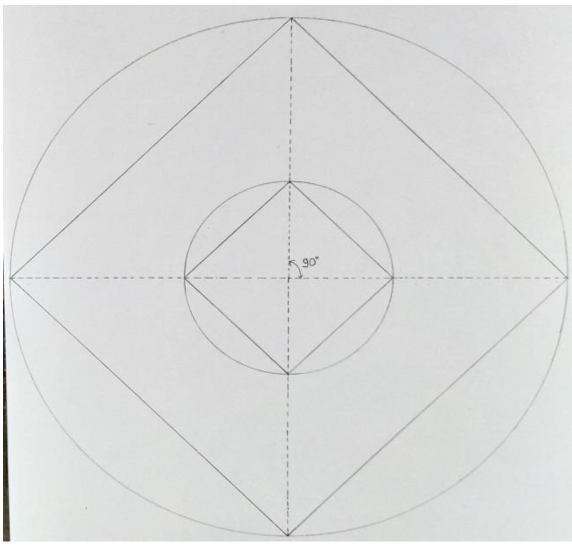


Figure-15: A big square is obtained by joining points of intersection of radial lines and the big circle by straight lines.

5.) Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta \tan\frac{\pi}{n}\right) = \frac{2\pi}{4} - 2\tan^{-1}\left(\cos 65^o \tan\frac{\pi}{4}\right) = 44.18^o$$

Make V-cut angle $\delta = 44.18^o$ at each vertex of small square such that δ is bisected by the radial line passing through that vertex (as shown in fig-16 below) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 10.18 cm (as shown in fig-17 below).

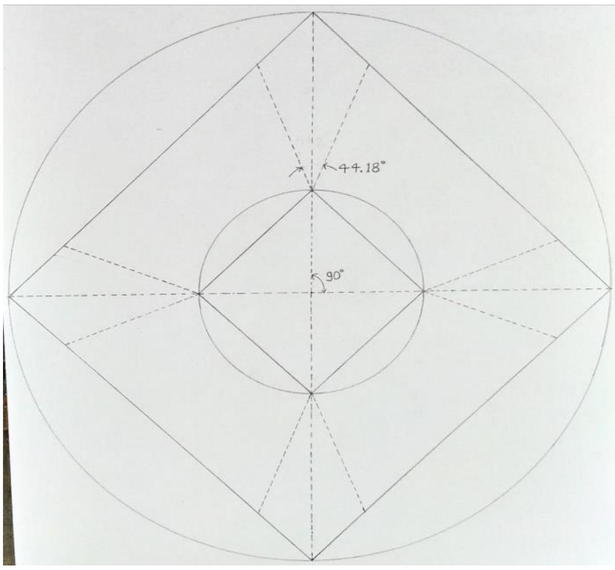


Figure-16: V-cut angle of 44.18^o is made at each vertex of small square

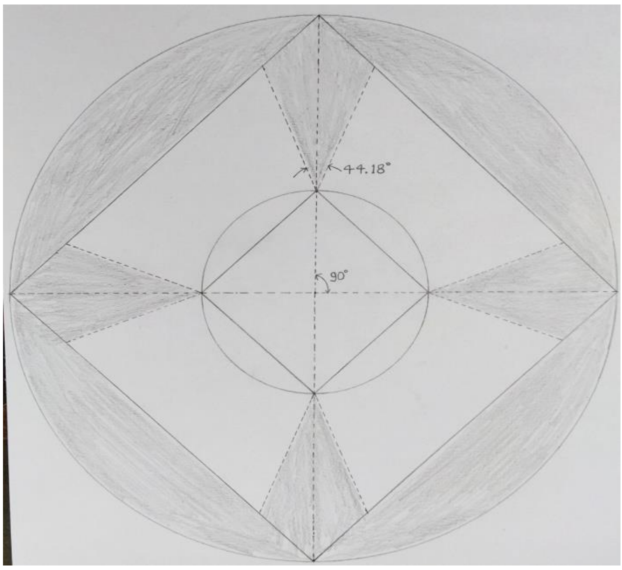


Figure-17: Undesired area (including V-parts) to be cut-removed from big circle is shaded

6) Cut & remove undesired area (as shaded in fig-17 above) from big circle (sheet) to get a blank of sheet (as shown in fig-18). Rotate the trapezoidal faces about the edges of small square (base) (as shown in fig-19)

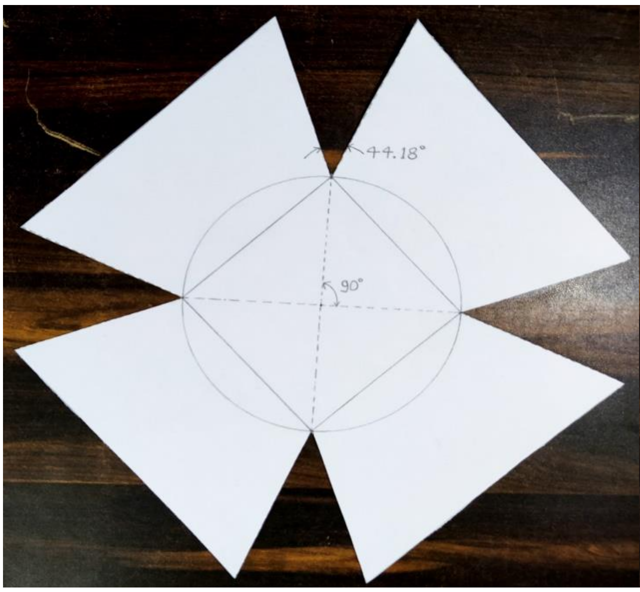


Figure-18: Blank of paper after cutting & removing undesired area (including V-parts) from big circle

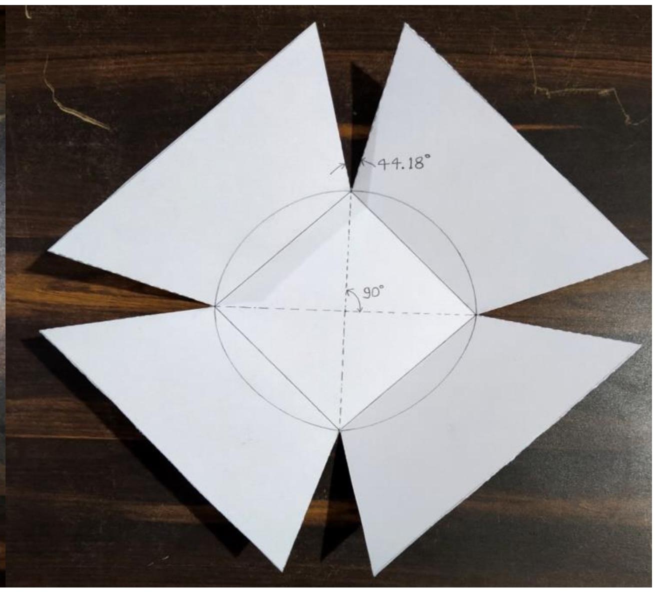


Figure-19: Rotating trapezoidal faces about the edges of square (base) which become lateral faces after meeting at the edges

7.) Rotate the trapezoidal faces about the edges of square (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired pyramidal flat container with square base (as shown in fig-20)

Figure-20: Desired pyramidal flat container after butt-joining the mating lateral edges using suitable adhesive

The desired pyramidal flat container with square base has some important parameters which are analytically calculated by using generalized formula (as derived above) as follows

1. Edge length (a_1) of square open end, is given as

$$a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n} = 5.4 + 2(4.5)\cos 65^o\ \tan\frac{\pi}{4} = 9.203\ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l\sqrt{1+\cos^2\theta\tan^2\frac{\pi}{n}} = 4.5\sqrt{1+\cos^2 65^o\tan^2\frac{\pi}{4}} = 4.885\ cm$$

3. Normal height (h), is given as

$$h = l\ \sin\theta = 4.5\ \sin 65^o = 4.078\ cm$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin 65^o\sin\frac{\pi}{4}\right) = 100.29^o$$

5. Total surface area (A_s) (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right) = \frac{1}{4}(4)\,(5.4)^2\cot\frac{\pi}{4} + 4(4.5)\left(5.4 + 4.5\cos 65^o\ \tan\frac{\pi}{4}\right)$$

$$= 160.592 cm^2$$

6. Volume (V), is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{4(4.5)\sin 65^o}{12}\left(3(5.4)^2\cot\frac{\pi}{4} + 4(4.5)^2\cos^2 65^o\tan\frac{\pi}{4} + 6(5.4)(4.5)\cos 65^o\right)$$

$$= 222.36\ cm^3$$

All above values of important parameters of pyramidal flat container with square base can be verified by accurate and precise measurements

2. Pyramidal flat container with regular hexagonal base: Let's make a paper model of pyramidal flat container with regular hexagonal base of each side 6 cm , slant height 3.5 cm and each trapezoidal lateral face is inclined with the plane of base at an angle 70^o.

Given data: $n =$ number of sides of regular hexagonal base $= 6$

$a =$ edge length of regular hexagonal base $= 6$ cm

$l =$ slant height of pyramidal flat container $= 3.5$ cm

$\theta =$ angle of inclination of each trapezoidal lateral face with plane of base $= 70^o$

Following steps (as shown in pictures below) are used to make a pyramidal flat container (of desired dimensions) with regular hexagonal base using a sheet of paper

1.) Calculate radius (r) of small circle

$$r = \frac{a}{2} cosec \frac{\pi}{n} = \frac{6}{2} cosec \frac{\pi}{6} = 6\ cm$$

Draw a circle with radius $r = 6\ cm$ on the sheet of paper (See fig-21 below)

2.) Calculate radius (R) of big circle

$$R = \frac{a}{2} cosec \frac{\pi}{n} + l \sec \frac{\pi}{n} = \frac{6}{2} cosec \frac{\pi}{6} + 3.5 \sec \frac{\pi}{6} = 10.04\ cm$$

Draw another circle of radius $R = 10.04\ cm$ concentric with small circle (See fig-22)

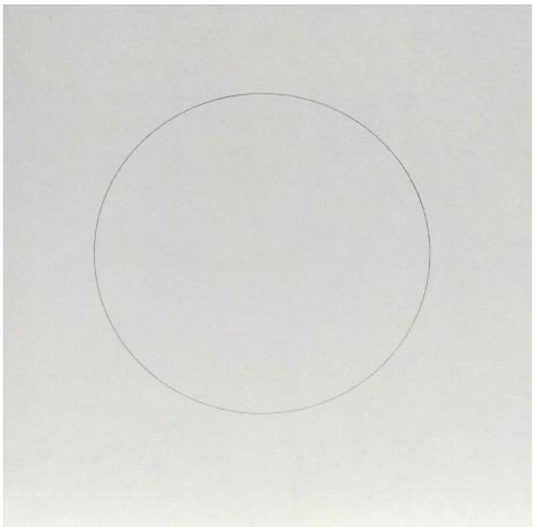


Figure-21: A circle of radius 6 cm is drawn on the sheet of paper

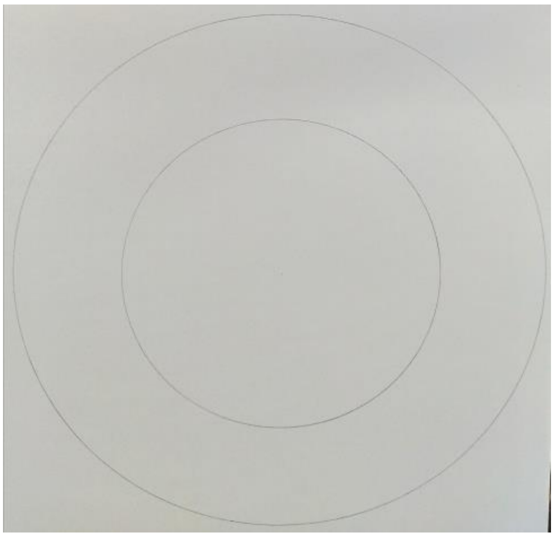


Figure-22: A big circle of radius 10.04 cm drawn on the sheet of paper is concentric with small circle of radius 6 cm

3.) Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{6} = 60^o$ & draw a regular hexagon of each side $a = 6\ cm$ circumscribed by the small circle of radius $r = 6\ cm$ (as shown in fig-23 below). Join the vertices of regular hexagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 10.04 cm (As shown in fig-24 below)

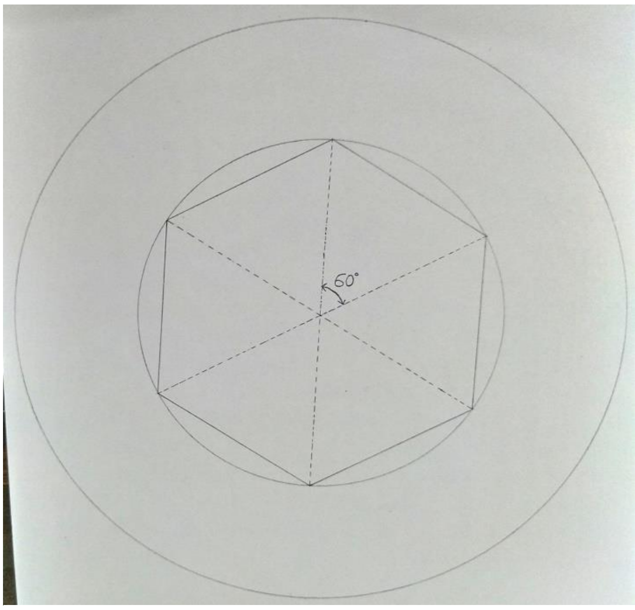


Figure-23: A regular hexagon of side6 cm drawn on the sheet of paper is circumscribed by small circle of radius 6 cm

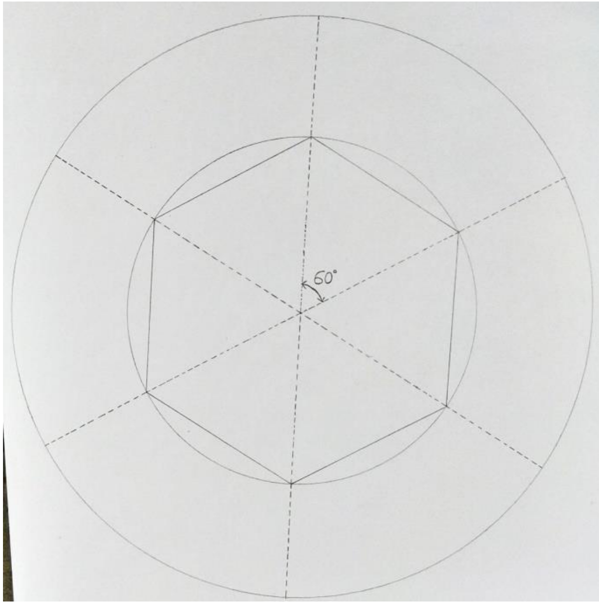


Figure-24: Dotted straight lines joining the vertices of regular hexagon to the centre are extended so as to intersect big circle of radius 10.04 cm

4.) Join the points of intersection of radial lines & big circle by the straight lines to get a big regular hexagon (See fig-25)

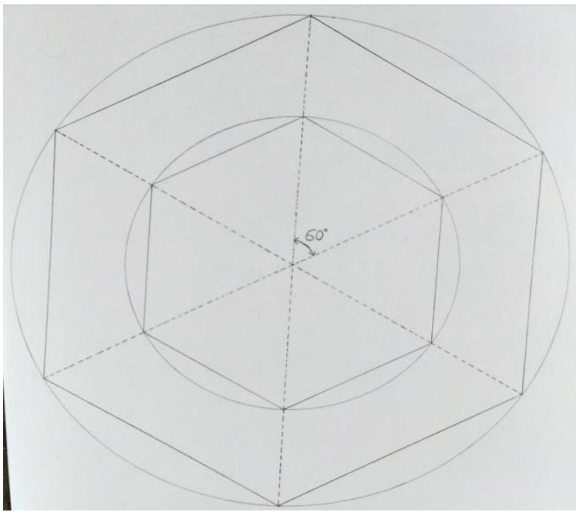


Figure-25: A big regular hexagon is obtained by joining points of intersection of radial lines and the big circle by straight lines.

5.) Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta \tan\frac{\pi}{n}\right) = \frac{2\pi}{6} - 2\tan^{-1}\left(\cos 70^o \tan\frac{\pi}{6}\right) = 37.66^o$$

Make V-cut angle $\delta = 37.66^o$ at each vertex of small regular hexagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-26 below) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 10.04 cm (as shown in fig-27 below).

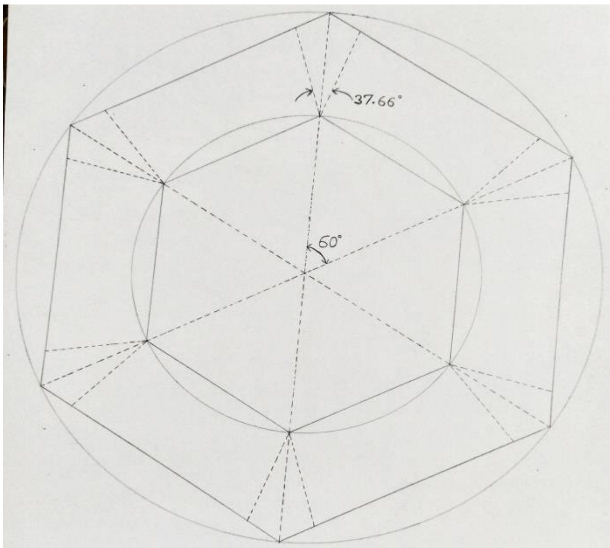


Figure-26: V-cut angle of 37.66^o is made at each vertex of small regular hexagon

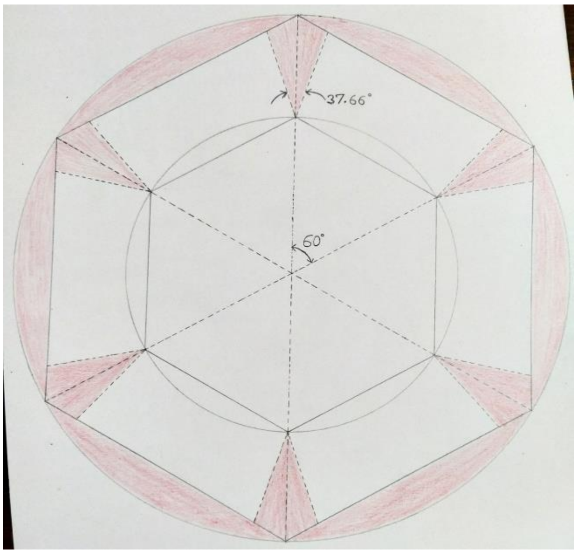


Figure-27: Undesired area (including V-parts) to be removed from big circle is shaded

6) Cut & remove undesired area (as shaded in fig-27 above) from big circle (sheet) to get a blank of paper sheet (as shown in fig-28 below). Rotate the trapezoidal faces about the edges of small regular hexagon (base) (as shown in fig-29 below)

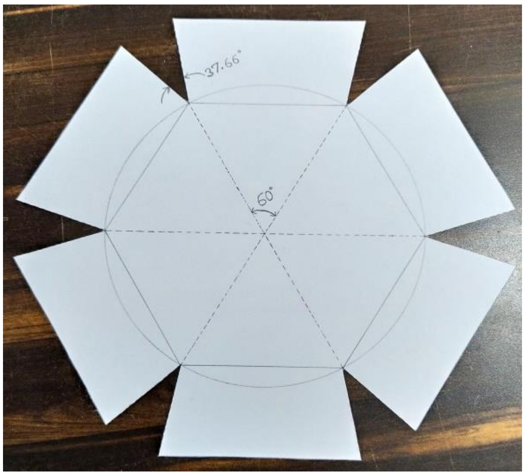


Figure-28: Blank of paper after cutting & removing undesired area (including V-parts) from big circle

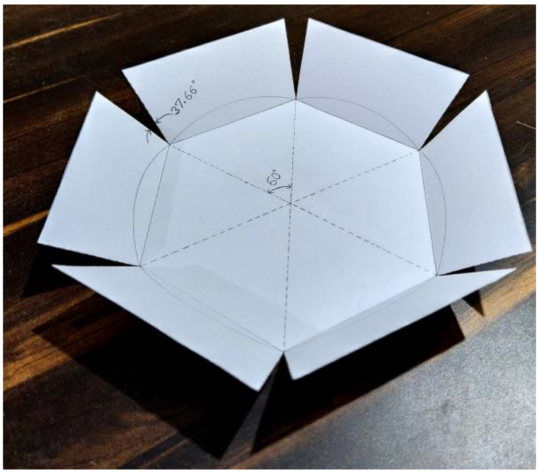


Figure-29: Rotating trapezoidal faces about the edges of regular hexagon (base) which become lateral faces after meeting at the edges

7.) Rotate the trapezoidal faces about the edges of regular hexagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired pyramidal flat container with regular hexagonal base (as shown in fig-30)

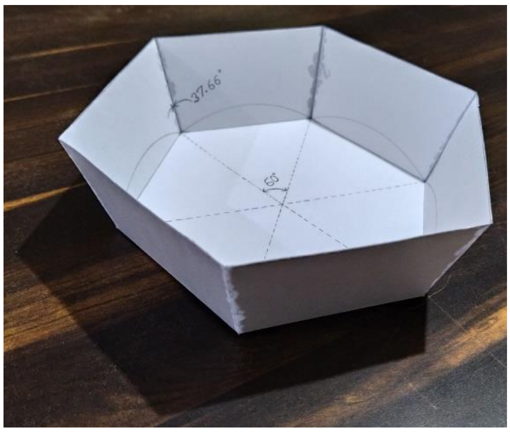


Figure-30: Desired pyramidal flat container after butt-joining the mating lateral edges using suitable adhesive

The desired pyramidal flat container with regular hexagonal base has some important parameters which are analytically calculated by using generalized formula (as derived earlier) as follows

1. Edge length (a_1) of regular hexagonal open end, is given as

$$a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n} = 6 + 2(3.5)\cos 70^o\ \tan\frac{\pi}{6} = 7.382\ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l\sqrt{1+\cos^2\theta\tan^2\frac{\pi}{n}} = 3.5\sqrt{1+\cos^2 70^o\tan^2\frac{\pi}{6}} = 3.567\ cm$$

3. Normal height (h), is given as

$$h = l\ \sin\theta = 3.5\ \sin 70^o = 3.289\ cm$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin 70^o\sin\frac{\pi}{6}\right) = 123.95^o$$

5. Total surface area (A_s) (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right) = \frac{1}{4}(6)\,(6)^2\cot\frac{\pi}{6} + 6(3.5)\left(6 + 3.5\cos 70^o\ \tan\frac{\pi}{6}\right)$$

$$= 234.044\ cm^2$$

6. Volume (V), is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{6(3.5)\sin 70^o}{12}\left(3(6)^2\cot\frac{\pi}{6} + 4(3.5)^2\cos^2 70^o\tan\frac{\pi}{6} + 6(6)(3.5)\cos 70^o\right)$$

$$= 383.925\ cm^3$$

All the values of parameters computed above can be verified by the accurate and precise measurements

3. Pyramidal flat container with regular pentagonal base: Let's make a paper model of pyramidal flat container with regular pentagonal base of each side 5 cm , slant height 4 cm and each trapezoidal lateral face is inclined with the plane of base at an angle 80^o.

Given data: n = number of sides of regular pentagonal base = 5

a = edge length of regular pentagonal base = 5 cm

l = slant height of pyramidal flat container = 4 cm

θ = angle of inclination of each trapezoidal lateral face with plane of base = 80^o

Following steps (as shown in pictures below) are used to make a pyramidal flat container (of desired dimensions) with regular pentagonal base using a sheet of paper

1.) Calculate radius (r) of small circle

$$r = \frac{a}{2} cosec \frac{\pi}{n} = \frac{5}{2} cosec \frac{\pi}{5} = 4.25\ cm$$

Draw a circle with radius $r = 4.25\ cm$ on the sheet of paper (See fig-31 below)

2.) Calculate radius (R) of big circle

$$R = \frac{a}{2} cosec \frac{\pi}{n} + l \sec\frac{\pi}{n} = \frac{5}{2} cosec \frac{\pi}{5} + 4 \sec\frac{\pi}{5} = 9.2\ cm$$

Draw another circle of radius $R = 9.2\ cm$ concentric with small circle (See fig-31 below)

3.) Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{5} = 72^o$ & draw a regular pentagon of each side $a = 5\ cm$ circumscribed by the small circle of radius $4.25\ cm$. Join the vertices of regular pentagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 9.2 cm (As shown in fig-31 below)

4.) Join the points of intersection of radial lines & big circle by the straight lines to get a big regular pentagon (See fig-31 below)

5.) Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta \tan\frac{\pi}{n}\right) = \frac{2\pi}{5} - 2\tan^{-1}\left(\cos 80^o \tan\frac{\pi}{5}\right) = 57.62^o$$

Make V-cut angle $\delta = 57.62^o$ at each vertex of small regular pentagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-31 below) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 9.2 cm (Follow the same procedure as mentioned above)

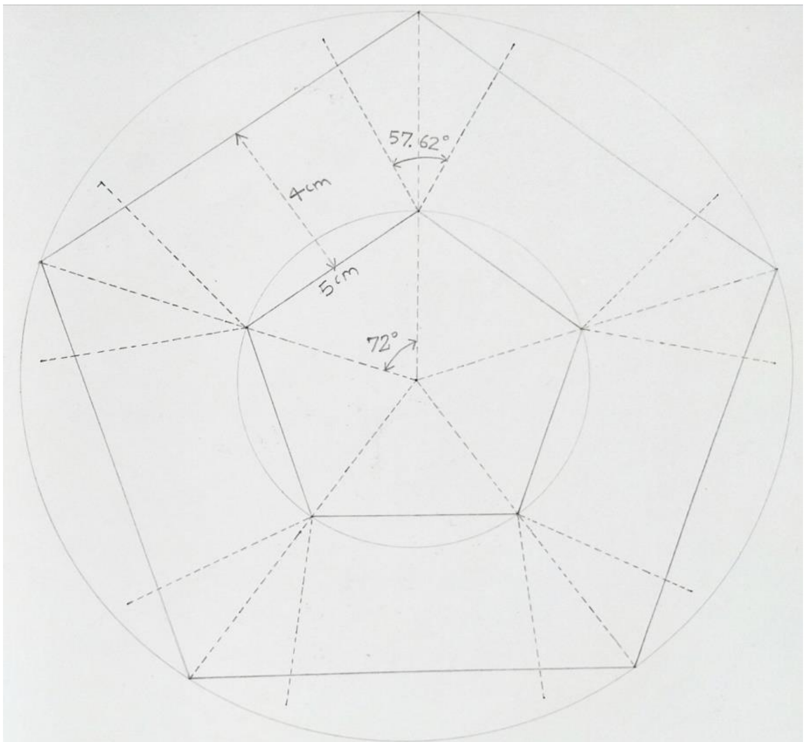


Figure-31: Complete drawing on the sheet of paper to make desired pyramidal flat container with regular pentagonal base from a circular sheet of radius 9.2 cm

6) Cut & remove undesired area (see fig-31 above) from big circle (sheet) to get a blank of paper sheet (as shown in fig-32 below). Rotate the trapezoidal faces about the edges of small regular pentagon (base) (as shown in fig-33 below)

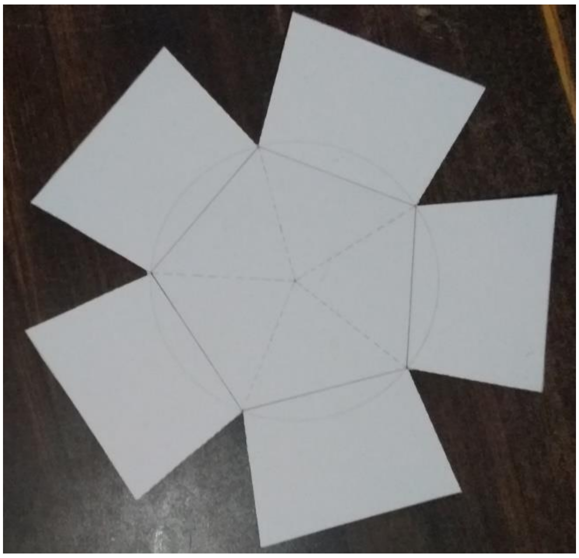


Figure-32: Blank of paper after cutting & removing undesired area (including V-parts) from big circle

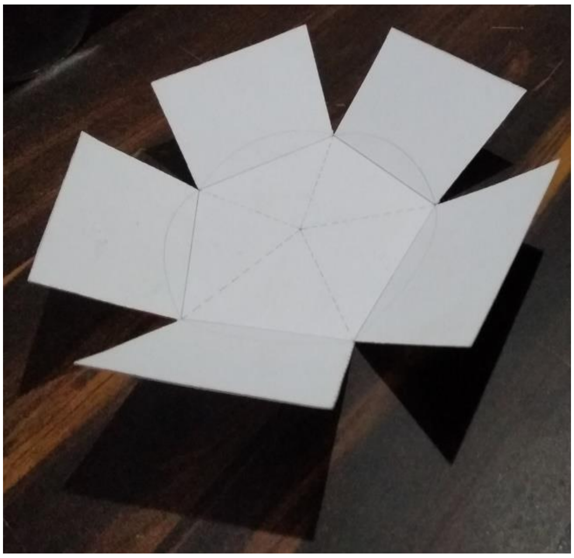


Figure-33: Rotating trapezoidal faces about the edges of regular pentagon (base) which become lateral faces after meeting at the edges

7.) Rotate the trapezoidal faces about the edges of regular pentagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired pyramidal flat container with regular pentagonal base (as shown in fig-34)

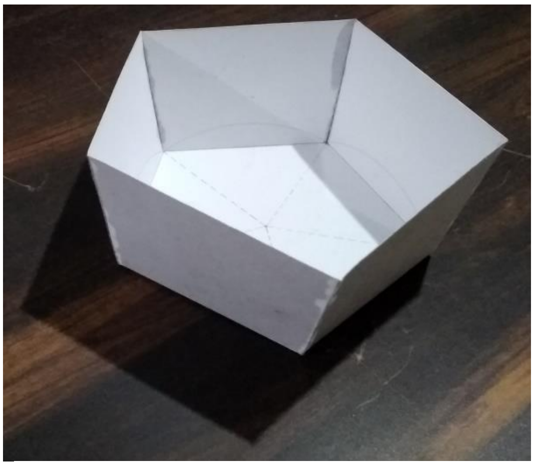


Figure-34: Desired pyramidal flat container after butt-joining the mating lateral edges using suitable adhesive

The desired pyramidal flat container with regular pentagonal base has some important parameters which are analytically calculated by using generalized formula (as derived earlier) as follows

1. Edge length (a_1) of regular pentagonal open end, is given as

$$a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n} = 5 + 2(4)\cos 80^o\ \tan\frac{\pi}{5} = 6\ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l\sqrt{1+\cos^2\theta\tan^2\frac{\pi}{n}} = 4\sqrt{1+\cos^2 80^o\tan^2\frac{\pi}{5}} = 4.032\ cm$$

3. Normal height $(\boldsymbol{h})$, is given as

$$h = l\ \sin\theta = 4\ \sin 80^o = 3.939\ cm$$

4. Dihedral angle $(\boldsymbol{\theta_d})$ between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin 80^o\sin\frac{\pi}{5}\right) = 109.26^o$$

5. Total surface area $(\boldsymbol{A_s})$ (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right) = \frac{1}{4}(5)\,(5)^2\cot\frac{\pi}{5} + 5(4)\left(5 + 4\cos 80^o\ \tan\frac{\pi}{5}\right)$$

$$= 153.105\ cm^2$$

6. Volume (V), is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{5(4)\sin 80^o}{12}\left(3(5)^2\cot\frac{\pi}{5} + 4(4)^2\cos^2 80^o\tan\frac{\pi}{5} + 6(5)(4)\cos 80^o\right)$$

$$= 205.937\ \ cm^3$$

All above values of important parameters of pyramidal flat container with regular pentagonal base can be verified by accurate and precise measurements.

4. Pyramidal flat container with regular heptagonal base: Let's make a paper model of pyramidal flat container with regular heptagonal base of each side $4\ cm$, slant height $4\ cm$ and each trapezoidal lateral face is inclined with the plane of base at an angle 110^o.

Given data: $n = \text{number of sides of regular heptagonal base} = 7$

$a = \text{edge length of regular heptagonal base} = 4\text{ cm}$

$l = \text{slant height of pyramidal flat container} = 4\text{ cm}$

$\theta = \text{angle of inclination of each trapezoidal lateral face with plane of base} = 110^o$

Following steps (as shown in pictures below) are used to make a pyramidal flat container (of desired dimensions) with regular heptagonal base using a sheet of paper

1.) Calculate radius (r) of small circle

$$r = \frac{a}{2}cosec\frac{\pi}{n} = \frac{4}{2}cosec\frac{\pi}{7} = 4.61\ cm$$

Draw a circle with radius $r = 4.61\ cm$ on the sheet of paper (See fig-35 below)

2.) Calculate radius (R) of big circle

$$R = \frac{a}{2} cosec\frac{\pi}{n} + l\sec\frac{\pi}{n} = \frac{4}{2} cosec\frac{\pi}{7} + 4\sec\frac{\pi}{7} = 9.05\ cm$$

Draw another circle of radius $R = 9.05\ cm$ concentric with small circle (See fig-35 below)

3.) Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{7} = 51.43^o$ & draw a regular heptagon of each side $a = 4\ cm$ circumscribed by the small circle of radius $4.61\ cm$. Join the vertices of regular heptagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 9.05 cm (As shown in fig-35 below)

4.) Join the points of intersection of radial lines & big circle by the straight lines to get a big regular heptagon (See fig-35 below)

5.) Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta \tan\frac{\pi}{n}\right) = \frac{2\pi}{7} - 2\tan^{-1}\left(\cos 110^o \tan\frac{\pi}{7}\right) = 70.13^o$$

Make V-cut angle $\delta = 70.13^o$ at each vertex of small regular heptagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-35 below) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 9.05 cm (Follow the same procedure as mentioned above)

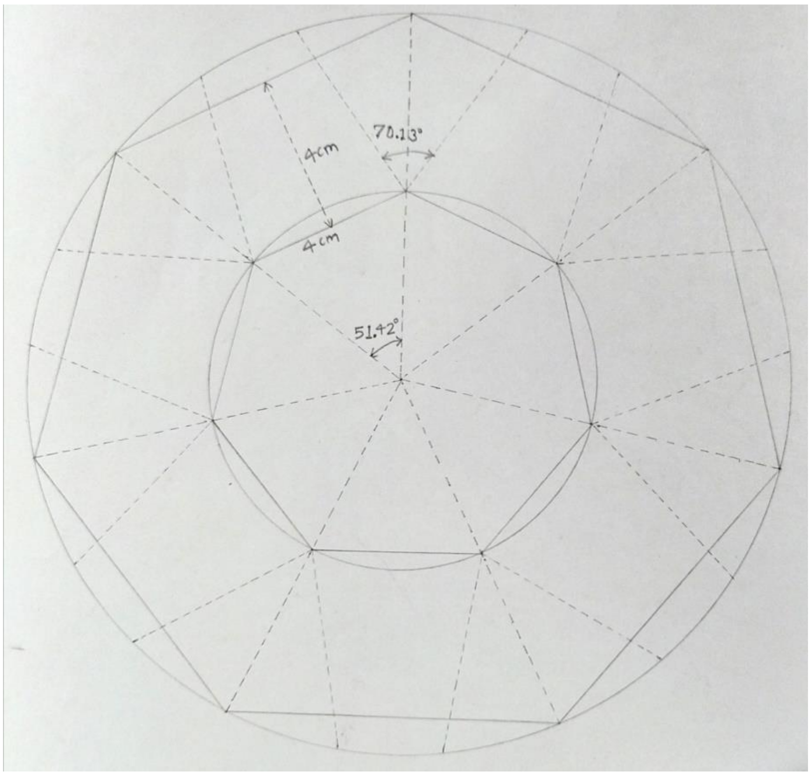


Figure-35: Complete drawing on the sheet of paper to make desired pyramidal flat container with regular heptagonal base from a circular sheet of radius 9.05 cm

6) Cut & remove undesired area (see fig-35 above) from big circle (sheet) to get a blank of paper sheet (as shown in fig-36 below). Rotate the trapezoidal faces about the edges of small regular heptagon (base) (as shown in fig-37 below)

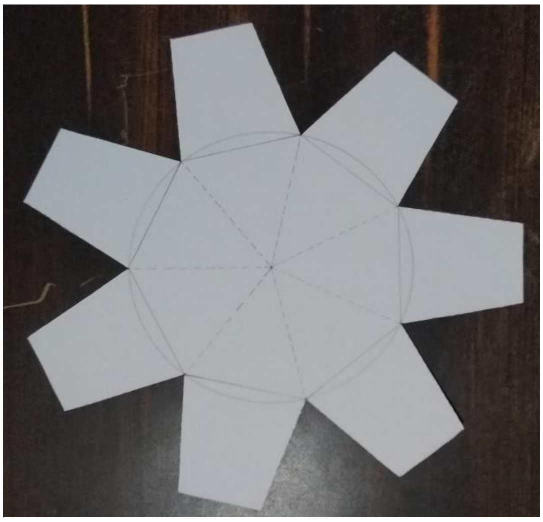


Figure-36: Blank of paper after cutting & removing undesired area (including V-parts) from big circle

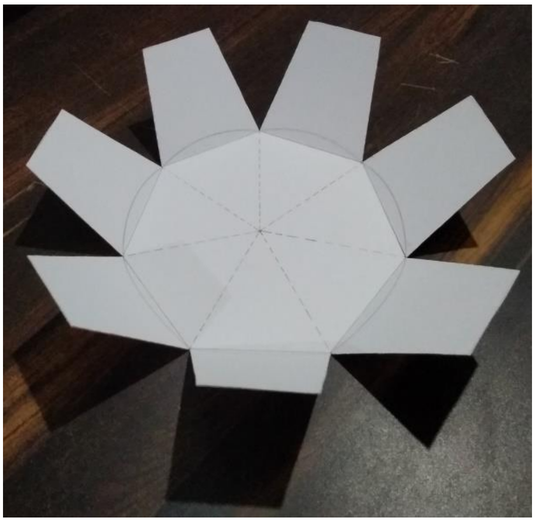


Figure-37: Rotating trapezoidal faces about the edges of regular heptagon (base) which become lateral faces after meeting at the edges

7.) Rotate the trapezoidal faces about the edges of regular heptagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired pyramidal flat container with regular heptagonal base (as shown in fig-38)

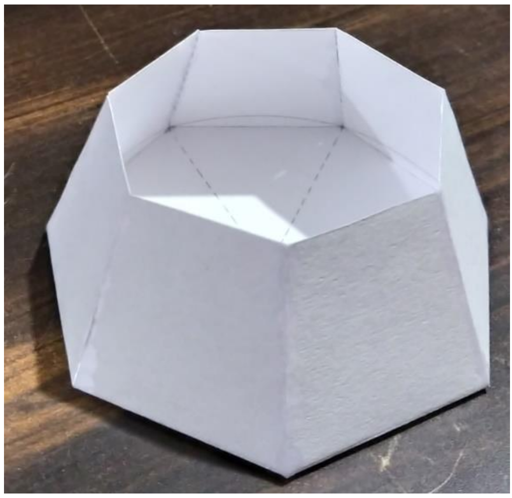


Figure-38: Desired pyramidal flat container after butt-joining the mating lateral edges using suitable adhesive

The desired pyramidal flat container with regular heptagonal base has some important parameters which are analytically calculated by using generalized formula (as derived earlier) as follows

1. Edge length (a_1) of regular heptagonal open end, is given as

$$a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n} = 4 + 2(4)\cos 110^o\ \tan\frac{\pi}{7} = 2.682\ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l\sqrt{1+\cos^2\theta \tan^2\frac{\pi}{n}} = 4\sqrt{1+\cos^2 110^o \tan^2\frac{\pi}{7}} = 4.054\ cm$$

3. Normal height (h), is given as

$$h = l\sin\theta = 4\sin 110^o = 3.759\ cm$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta \sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin 110^o \sin\frac{\pi}{7}\right) = 131.88^o$$

5. Total surface area (A_s) (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right) = \frac{1}{4}(7)\,(4)^2\cot\frac{\pi}{7} + 7(4)\left(4 + 4\cos 110^o\ \tan\frac{\pi}{7}\right)$$

$$= 151.695\ cm^2$$

6. Volume (V), is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{7(4)\sin 110^o}{12}\left(3(4)^2\cot\frac{\pi}{7} + 4(4)^2\cos^2 110^o\tan\frac{\pi}{7} + 6(4)(4)\cos 110^o\right)$$

$$= 154.458\ cm^3$$

All above values of important parameters of pyramidal flat container with regular heptagonal base can be verified by accurate and precise measurements.

5. Pyramidal flat container with regular octagonal base: Let's make a paper model of pyramidal flat container with regular octagonal base of each side $4\ cm$, slant height $4\ cm$ and each trapezoidal lateral face is inclined with the plane of base at an angle 60^o.

Given data: $n = \text{number of sides of regular octagonal base} = 8$

$a = \text{edge length of regular octagonal base} = 4\ \text{cm}$

$l = \text{slant height of pyramidal flat container} = 4\ \text{cm}$

$\theta = \text{angle of inclination of each trapezoidal lateral face with plane of base} = 60^o$

Following steps (as shown in pictures below) are used to make a pyramidal flat container (of desired dimensions) with regular octagonal base using a sheet of paper

1.) Calculate radius (r) of small circle

$$r = \frac{a}{2}cosec\frac{\pi}{n} = \frac{4}{2}cosec\frac{\pi}{8} = 5.23\ cm$$

Draw a circle with radius $r = 5.23\ cm$ on the sheet of paper (See fig-39 below)

2.) Calculate radius (R) of big circle

$$R = \frac{a}{2} cosec\frac{\pi}{n} + l\sec\frac{\pi}{n} = \frac{4}{2} cosec\frac{\pi}{8} + 4\sec\frac{\pi}{8} = 9.55\ cm$$

Draw another circle of radius $R = 9.55\ cm$ concentric with small circle (See fig-39 below)

3.) Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{8} = 45^o$ & draw a regular octagon of each side $a = 4\ cm$ circumscribed by the small circle of radius $5.23\ cm$. Join the vertices of regular octagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 9.55 cm (As shown in fig-39 below)

4.) Join the points of intersection of radial lines & big circle by the straight lines to get a big regular octagon (See fig-39 below)

5.) Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right) = \frac{2\pi}{8} - 2\tan^{-1}\left(\cos 60^o\tan\frac{\pi}{8}\right) = 21.6^o$$

Make V-cut angle $\delta = 21.6^o$ at each vertex of small regular octagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-39 below) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 9.55 cm (Follow the same procedure as mentioned above)

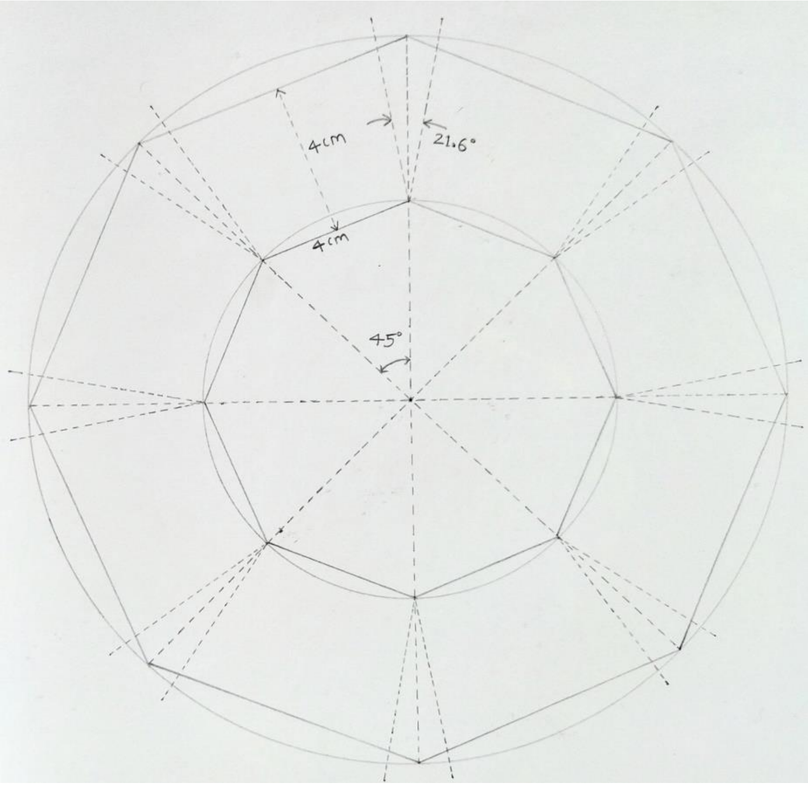


Figure-39: Complete drawing on the sheet of paper to make desired pyramidal flat container with regular octagonal base from a circular sheet of radius 9.55 cm

6) Cut & remove undesired area (see fig-39 above) from big circle (sheet) to get a blank of paper sheet (as shown in fig-40 below). Rotate the trapezoidal faces about the edges of small regular octagon (base) (as shown in fig-41 below)

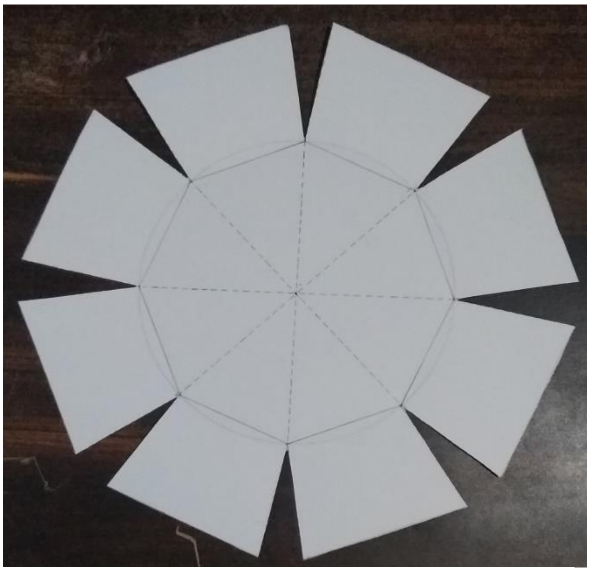


Figure-40: Blank of paper after cutting & removing undesired area (including V-parts) from big circle

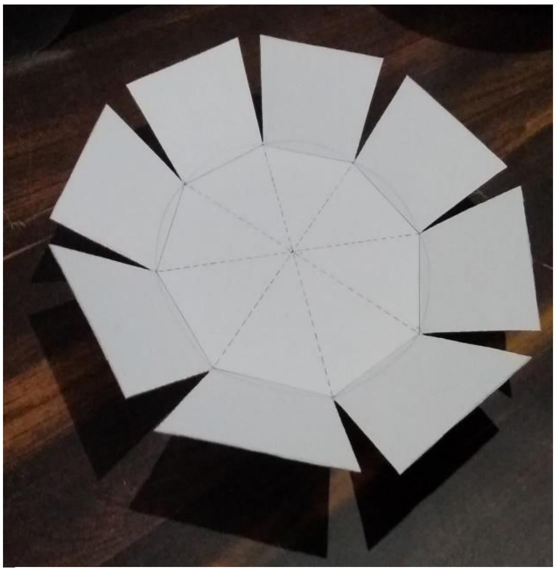


Figure-41: Rotating trapezoidal faces about the edges of regular octagon (base) which become lateral faces after meeting at the edges

7.) Rotate the trapezoidal faces about the edges of regular octagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired pyramidal flat container with regular octagonal base (as shown in fig-42)

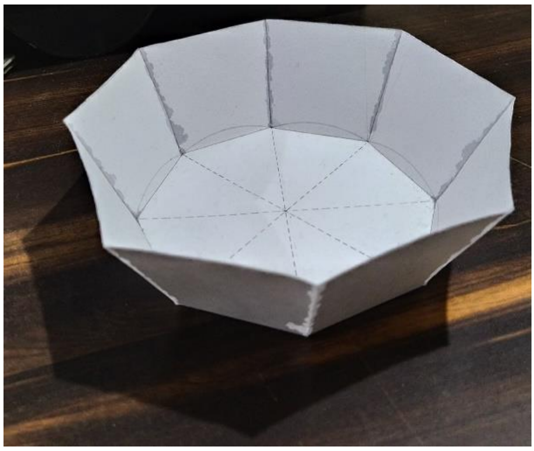


Figure-42: Desired pyramidal flat container after butt-joining the mating lateral edges using suitable adhesive

The desired pyramidal flat container with regular octagonal base has some important parameters which are analytically calculated by using generalized formula (as derived earlier) as follows

1. Edge length (a_1) of regular octagonal open end, is given as

$$a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n} = 4 + 2(4)\cos 60^o\ \tan\frac{\pi}{8} = 5.657\ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l\sqrt{1+\cos^2\theta\tan^2\frac{\pi}{n}} = 4\sqrt{1+\cos^2 60^o\tan^2\frac{\pi}{8}} = 4.085\ cm$$

3. Normal height $(\boldsymbol{h})$, is given as

$$h = l\sin\theta = 4\sin 60^o = 3.464\ cm$$

4. Dihedral angle $(\boldsymbol{\theta_d})$ between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin 60^o\sin\frac{\pi}{8}\right) = 141.29^o$$

5. Total surface area $(\boldsymbol{A_s})$ (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right) = \frac{1}{4}(8)\,(4)^2\cot\frac{\pi}{8} + 8(4)\left(4 + 4\cos 60^o\ \tan\frac{\pi}{8}\right)$$

$$= 231.764\ cm^2$$

6. Volume (V), is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{8(4)\sin 60^o}{12}\left(3(4)^2\cot\frac{\pi}{8} + 4(4)^2\cos^2 60^o\tan\frac{\pi}{8} + 6(4)(4)\cos 60^o\right)$$

$$= 393.775\ \ cm^3$$

The area of circular sheet of radius 9.55 cm removed as scrap, is equal to the difference of area of circular sheet and the total surface area of container which is given as

$$= \pi(9.55)^2 - 231.764$$

$$= 54.757\ cm^2$$

The percentage (%) of total area of circular sheet used to make pyramidal flat container with regular octagonal base is

$$= \frac{\text{Total surface area of container}}{\text{Area of circular sheet of radius 9.55 cm}} \times 100\ \%$$

$$= \frac{231.764}{\pi(9.55)^2} \times 100\ \%$$

$$= 80.89\ \%$$

All above values of important parameters of pyramidal flat container with regular octagonal base can be verified by accurate and precise measurements.

So far, we have derived generalized formula & discussed in details the method to make pyramidal flat container with regular n-gonal base. Applying the same approach & method as in case of pyramidal flat container, we will derive generalized formula & discuss in details the method to make right pyramid with regular n-gonal base.

Right pyramid (shell) with regular n-gonal base: We know that all the triangular lateral faces of a right pyramid intersect each other at the apex. If we rotate all the trapezoidal lateral faces of a pyramidal flat container about the edges of its regular n-gonal base through an equal angle in such a way that all the lateral faces meet at a single point (i.e. apex) then a pyramidal flat container (having slant height l & edge length of regular n-gonal base a and an angle θ (measured externally) of inclination of each lateral face with the plane of base) becomes a right pyramid **(**having slant height l, regular n-gonal base of edge length a & an angle θ (measured externally) of inclination of each lateral face with the plane of base). In this case, each trapezoidal lateral face of pyramidal flat container becomes an isosceles triangular lateral face of right pyramid (See fig-43 below)

Angle of inclination (θ) of each isosceles triangular lateral face with the plane of regular n-gonal base of right pyramid:

We know that edge length (a_1) of regular n-gonal open end of pyramidal flat container is given as

$$a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n}$$

Now, in order to make all the trapezoidal lateral faces of flat container meet at a point, rotate all the lateral faces about the edges of its base through an angle θ in such a way that each trapezoidal lateral face of pyramidal flat container becomes an isosceles triangular lateral face of right pyramid i.e. the edge length a_1 of trapezoidal lateral face must be zero i.e. $a_1 = 0$ (as shown in fig-43)

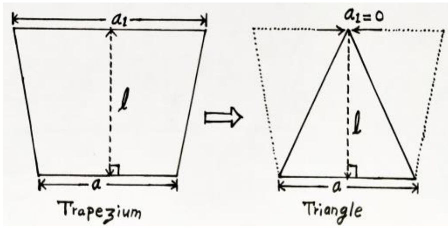


Figure-43: A trapezoidal lateral face of flat container becomes a triangular lateral face of right pyramid when edge length $a_1 = 0$

$$\therefore\ a + 2l\cos\theta\ \tan\frac{\pi}{n} = 0$$

$$2l\cos\theta\ \tan\frac{\pi}{n} = -a$$

$$\cos\theta\ = \frac{-a}{2l\tan\frac{\pi}{n}}$$

$$\cos\theta\ = \frac{-a}{2l}\ \cot\frac{\pi}{n} \qquad \left(-\,ve \text{ sign indicates that } \frac{\pi}{2} < \theta < \pi\right)$$

$$\theta\ = \pi - \cos^{-1}\left(\frac{a}{2l}\cot\frac{\pi}{n}\right)$$

Hence, the **angle of inclination of each isosceles triangular lateral face (θ), measured externally with the plane of regular n-gonal base with each side a in a right pyramid of slant height l,** is given as

$$\boldsymbol{\theta\ = \pi - \cos^{-1}\left(\frac{a}{2l}\cot\frac{\pi}{n}\right)}$$

The above formula is very important for computing the angle of inclination of isosceles triangular lateral face with the plane of regular n-gonal base of right pyramid when number of sides n & edge length a of regular n-gonal base and slant height l are known.

V-cut angle (δ) required to make right pyramid with regular n-gonal base from a circular sheet:

Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in generalized formula of V-cut angle for pyramidal flat container with regular n-gonal base, V-cut angle δ required to cut remove V-parts & rotate each two co-planar planes about their intersecting straight edges i.e. edges of regular n-gonal base of right pyramid is obtained as follows

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right)$$

$$= \frac{2\pi}{n} - 2\tan^{-1}\left(\frac{-a}{2l}\cot\frac{\pi}{n}\tan\frac{\pi}{n}\right) \quad \text{(setting the value of } \cos\theta\text{)}$$

$$= \frac{2\pi}{n} - 2\tan^{-1}\left(\frac{-a}{2l}\right)$$

$$= \frac{2\pi}{n} + 2\tan^{-1}\left(\frac{a}{2l}\right)$$

Hence, V-cut angle δ required to cut remove V-parts & rotate each two co-planar planes about their intersecting straight edges i.e. edges of regular n-gonal base of right pyramid, is given as

$$\boldsymbol{\delta = \frac{2\pi}{n} + 2\tan^{-1}\left(\frac{a}{2l}\right)}$$

Length (L) of each lateral edge of right pyramid: Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in generalized formula for length of lateral edge of pyramidal flat container, the length of each lateral edge of a right pyramid with regular n-gonal base, is obtained as follows

$$L = l\sqrt{1 + \cos^2\theta\tan^2\frac{\pi}{n}}$$

$$= l\sqrt{1 + \left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2\tan^2\frac{\pi}{n}} \quad \text{(setting the value of } \cos\theta\text{)}$$

$$= l\sqrt{1 + \frac{a^2}{4l^2}\cot^2\frac{\pi}{n}\tan^2\frac{\pi}{n}}$$

$$= l\sqrt{1 + \frac{a^2}{4l^2}}$$

$$= \sqrt{l^2 + \left(\frac{a}{2}\right)^2}$$

Above result can also be derived by Pythagorean Theorem.

Hence, the **length (L) of each lateral edge of a right pyramid with slant height l & regular n-gonal base of each side a**, is given as

$$\boldsymbol{L = \sqrt{l^2 + \left(\frac{a}{2}\right)^2}}$$

Normal height (h) of right pyramid: Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in generalized formula for normal height of pyramidal flat container with regular n-gonal base, the normal height of a right pyramid with regular n-gonal base, is obtained as follows

$$h = l\sin\theta = l\sqrt{1-\cos^2\theta}$$

$$= l\sqrt{1-\left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2} \qquad \text{(setting the value of } \cos\theta)$$

$$= l\sqrt{1-\frac{a^2}{4l^2}\cot^2\frac{\pi}{n}}$$

$$= \sqrt{l^2-\frac{a^2}{4}\cot^2\frac{\pi}{n}}$$

Hence, the **normal height (h) of a right pyramid with slant height l & regular n-gonal base of each side a**, is given as

$$\boldsymbol{h = \sqrt{l^2-\frac{a^2}{4}\cot^2\frac{\pi}{n}}}$$

Dihedral angle (θ_d) between any two consecutive lateral faces of right pyramid: Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in generalized formula of dihedral angle for pyramidal flat container with regular n-gonal base, the dihedral angle between any two consecutive lateral faces of a right pyramid with regular n-gonal base, is obtained as follows

$$\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right)$$

$$= 2\cos^{-1}\left(\sin\frac{\pi}{n}\sqrt{1-\cos^2\theta}\right)$$

$$= 2\cos^{-1}\left(\sin\frac{\pi}{n}\sqrt{1-\left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2}\right) \qquad \text{(setting the value of } \cos\theta)$$

$$= 2\cos^{-1}\left(\sin\frac{\pi}{n}\sqrt{1-\frac{a^2}{4l^2}\cot^2\frac{\pi}{n}}\right)$$

$$= 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{n}-\frac{a^2}{4l^2}\cos^2\frac{\pi}{n}}\right)$$

Hence, the **dihedral angle (θ_d) between any two adjacent lateral faces of a right pyramid with slant height l & regular n-gonal base of each side a**, is given as

$$\boldsymbol{\theta_d = 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{n}-\frac{a^2}{4l^2}\cos^2\frac{\pi}{n}}\right)}$$

Total surface area (A_s) of right pyramid (base + lateral surface): Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in generalized formula for total surface area of pyramidal flat container with regular n-gonal base, the total surface area of a right pyramid with regular n-gonal base, is obtained as follows

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right)$$

$$= \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)\tan\frac{\pi}{n}\right) \qquad \text{(setting the value of}\cos\theta)$$

$$= \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a - \frac{a}{2}\right)$$

$$= \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + n\left(\frac{1}{2}al\right)$$

$$= \text{Area of regular polygonal base} + \text{Area of lateral surface}$$

Hence, the **total surface area ($\boldsymbol{A_s}$) (base + lateral surface) of a right pyramid with slant height $\boldsymbol{l}$ & regular n-gonal base of each side $\boldsymbol{a}$**, is given as

$$\boldsymbol{A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + n\left(\frac{1}{2}al\right)}$$

Volume (V) of right pyramid: Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in generalized formula for volume of pyramidal flat container with regular n-gonal base, the volume of a right pyramid with regular n-gonal base, is obtained as follows

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{nl}{12}\sqrt{1-\cos^2\theta}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{nl}{12}\sqrt{1-\left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2}\left(3a^2\cot\frac{\pi}{n} + 4l^2\left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2\tan\frac{\pi}{n} + 6al\left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)\right)$$

$$= \frac{nl}{12}\sqrt{1-\frac{a^2}{4l^2}\cot^2\frac{\pi}{n}}\left(3a^2\cot\frac{\pi}{n} + a^2\cot\frac{\pi}{n} - 3a^2\cot\frac{\pi}{n}\right)$$

$$= \frac{n}{12}\sqrt{l^2-\frac{a^2}{4}\cot^2\frac{\pi}{n}}\left(a^2\cot\frac{\pi}{n}\right)$$

$$= \frac{1}{3}\times\frac{1}{4}na^2\cot\frac{\pi}{n}\sqrt{l^2-\frac{a^2}{4}\cot^2\frac{\pi}{n}}$$

$$= \frac{1}{3}\times(\text{Area of regular polygonal base})\times(\text{Normal height})$$

Hence, the **volume (V) of a right pyramid with slant height $\boldsymbol{l}$ & regular n-gonal base of each side $\boldsymbol{a}$**, is given as

$$\boldsymbol{V = \frac{1}{12}na^2\cot\frac{\pi}{n}\sqrt{l^2-\frac{a^2}{4}\cot^2\frac{\pi}{n}}}$$

General steps, for making a right pyramid having regular n-gonal base of each side a, slant height l, using a thin sheet of paper, polymer, metal or alloy of desired thickness which can easily be cut, bent and butt-joined at mating edges, are the same as that of making a pyramidal flat container (as discussed above in details) but there is also an alternative & easier way to make a right pyramid with desired dimensions as follows

Step 1: Draw a circle with radius $r = \frac{a}{2} cosec \frac{\pi}{n}$ & centre 'O' on the sheet of desired material and thickness.

Step 2: Draw another circle of radius $R = \frac{a}{2} cosec \frac{\pi}{n} + l \sec \frac{\pi}{n}$ concentric with small circle of radius r on the sheet

Step 3: Take aperture angle $\frac{2\pi}{n}$ & draw a regular polygon $A_1A_2A_3 \dots \dots . A_{n-1}A_n$ with n no. of sides each of length a circumscribed by the small circle of radius r (as shown in fig-44 below)

Step 4: Join the vertices $A_1, A_2, A_3, \dots \dots . A_{n-1}$ & A_n to the centre O by dotted straight lines which when extended intersect big circle at the points $B_1, B_2, B_3 \dots \dots . B_{n-1}$ & B_n. Joint these points by straight lines to get a big regular n-gon $B_1B_2B_3 \dots \dots . B_{n-1}B_n$ (as shown in fig-44 below)

Step 5: Make V-cut angle $\delta = \frac{2\pi}{n} + 2\tan^{-1}\left(\frac{a}{2l}\right)$ at each vertex of small regular n-gon such that δ is bisected by the radial line passing through that vertex (as shown in fig-45) or

alternatively join the mid-point of each side of big regular n-gon to two closest vertices of small regular n-gon to get the same drawing

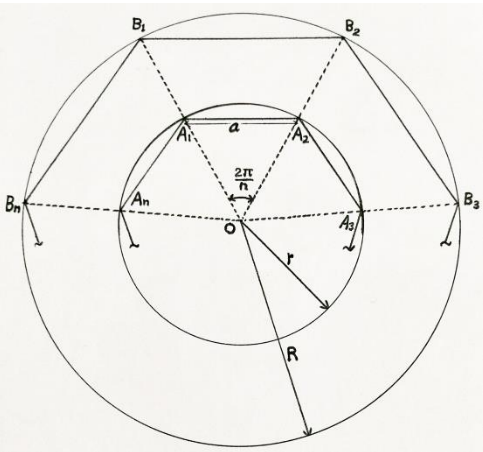


Figure-44: Small regular n-gon is circumscribed by circle of radius r & big regular n-gon is obtained by joining points of intersection of extended radial lines and the big circle.

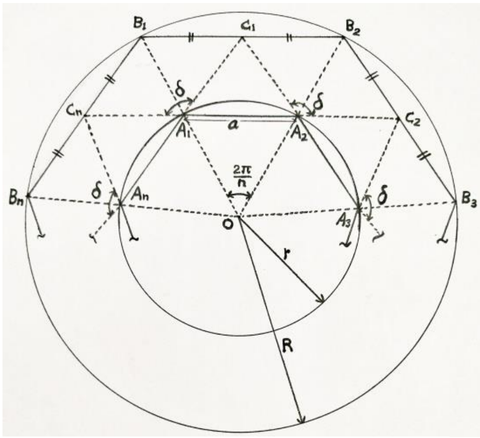


Figure-45: V-cut angle is made at each vertex of small regular n-gon to cut remove undesired area from big circle of radius R

Step 6: Mark V-cut parts & undesired area (as shaded in the fig-46 below) & cut remove undesired area from big circular sheet of radius R to get a blank of sheet (as shown in the fig-47 below)

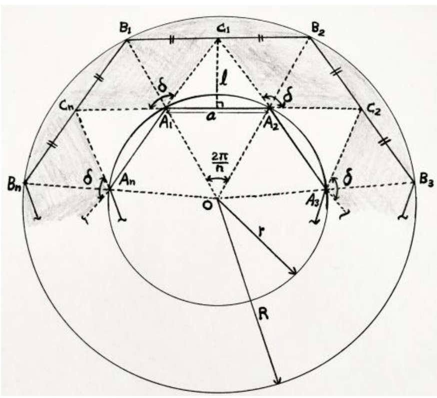


Figure-46: Undesired area to be cut removed is shaded

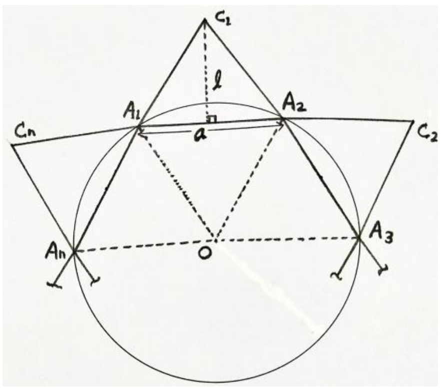


Figure-47: Blank of sheet after cut-removing undesired area

Step 7: Rotate or bend each of triangular faces about the edges of regular n-gonal base $A_1A_2A_3 \dots \dots . A_{n-1}A_n$ until their new edges (generated after cutting V-parts) coincide with one another & meet at apex. Then butt-join the mating lateral edges using a suitable adhesive or welding process to get a desired right pyramid

Thus, a desired right pyramid (as obtained above) with regular n-gonal base of each side a & slant height l, has the following important parameters

1. Length (L) of each lateral edge of right pyramid, is given as

$$L = \sqrt{l^2 + \left(\frac{a}{2}\right)^2}$$

2. Normal height (h) of right pyramid, is given as

$$h = \sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}}$$

3. Angle of inclination (θ) of each isosceles triangular lateral face with the plane of regular n-gonal base ,is given as

$$\theta = \pi - \cos^{-1}\left(\frac{a}{2l}\cot\frac{\pi}{n}\right)$$

4. Dihedral angle (θ_d) between any two consecutive lateral faces of right pyramid, is given as

$$\theta_d = 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{n} - \frac{a^2}{4l^2}\cos^2\frac{\pi}{n}}\right)$$

5. Total surface area (A_s) of right pyramid (base + lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + n\left(\frac{1}{2}al\right)$$

6. Volume (V) of right pyramid, is given as

$$V = \frac{1}{12}na^2\cot\frac{\pi}{n}\sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}}$$

We will now make a paper-model of right pyramid using general steps as mentioned above & the generalized formula as derived above.

Right pyramid (shell) with regular octagonal base: Let's make a paper model of right pyramid with regular octagonal base of each side 3 cm & slant height 5 cm

Given data: $n = \text{number of sides of regular octagonal base} = 8$

$$a = \text{edge length of regular octagonal base} = 3\text{ cm}$$

$$l = \text{slant height of right pyramid} = 5\text{ cm}$$

Following steps (as shown in pictures below) are used to make a right pyramid (of desired dimensions) with regular octagonal base using a sheet of paper

1.) Calculate radius (r) of small circle

$$r = \frac{a}{2} cosec \frac{\pi}{n} = \frac{3}{2} cosec \frac{\pi}{8} = 3.92\ cm$$

Draw a circle with radius $r = 3.92\ cm$ on the sheet of paper (See fig-48)

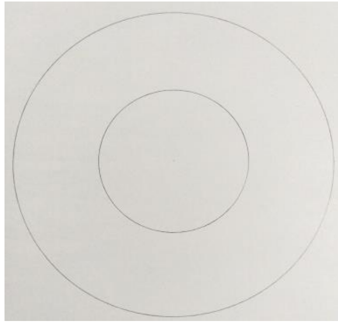


Figure-48: Two concentric circles of radii 3.92 cm & 9.33 cm are drawn on the sheet of paper

2.) Calculate radius (R) of big circle

$$R = \frac{a}{2} cosec \frac{\pi}{n} + l \sec\frac{\pi}{n} = \frac{3}{2} cosec \frac{\pi}{8} + 5 \sec\frac{\pi}{8} = 9.33\ cm$$

Draw another circle of radius $R = 9.33\ cm$ concentric with small circle (See fig-48)

3.) Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{8} = 45^o$ & draw a regular octagon of each side $a = 3\ cm$ circumscribed by the small circle of radius 3.92 cm (As shown in fig-49 below) . Join the vertices of regular octagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 9.33 cm (As shown in fig-50 below)

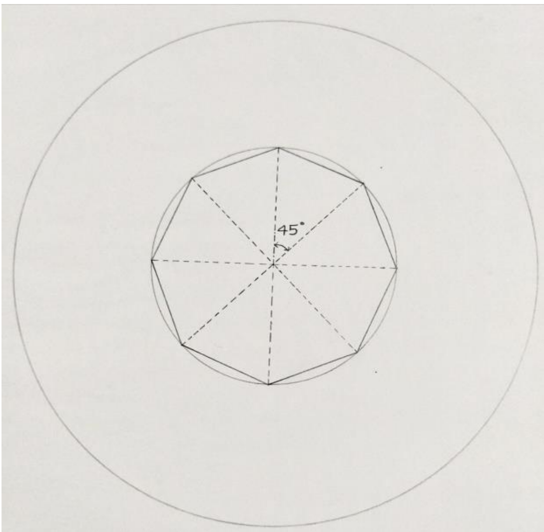


Figure-49: A regular octagon of side 3 cm drawn on the sheet of paper is circumscribed by small circle of radius 3.92 cm

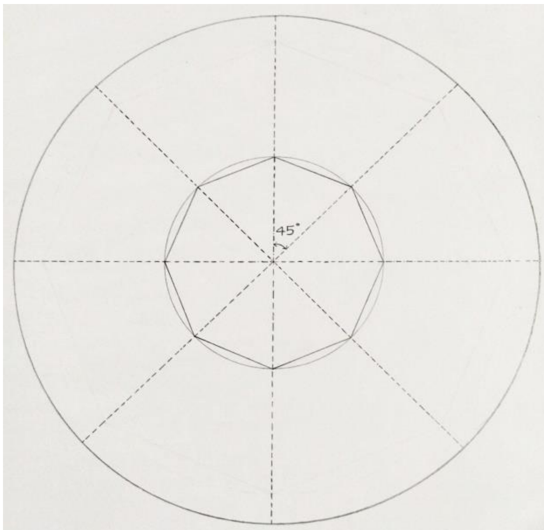


Figure-50: Dotted straight lines joining the vertices of regular octagon to the centre are extended so as to intersect big circle of radius 9.33 cm

4.) Join the points of intersection of radial lines & big circle by the straight lines to get a big regular octagon (See fig-51 below)

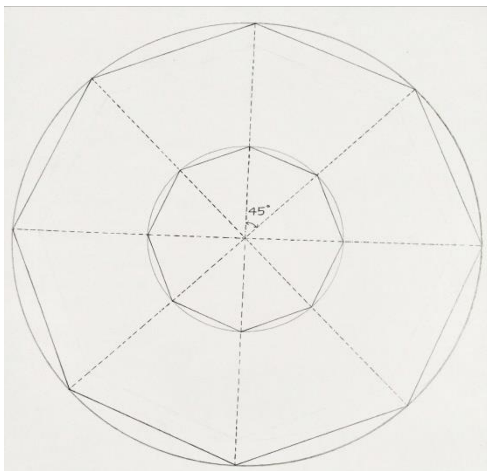


Figure-51: A big regular octagon is obtained by joining points of intersection of radial lines and the big circle by straight lines.

5.) Calculate V-cut angle (δ)

$$\delta=\frac{2\pi}{n}+2\tan^{-1}\left(\frac{a}{2l}\right)=\frac{2\pi}{8}+2\tan^{-1}\left(\frac{3}{2\times 5}\right)=78.4^{o}$$

Make V-cut angle $\delta=78.4^{o}$ at each vertex of small regular octagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-52 below) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 9.33 cm (as shown in fig-53 below)

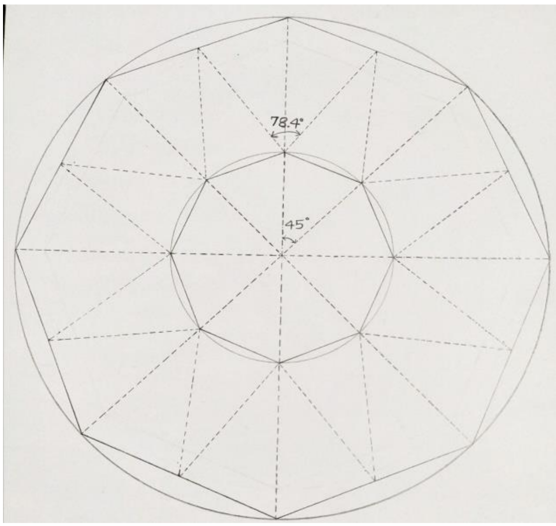


Figure-52: V-cut angle of 78.4^{o} is made at each vertex of small regular octagon

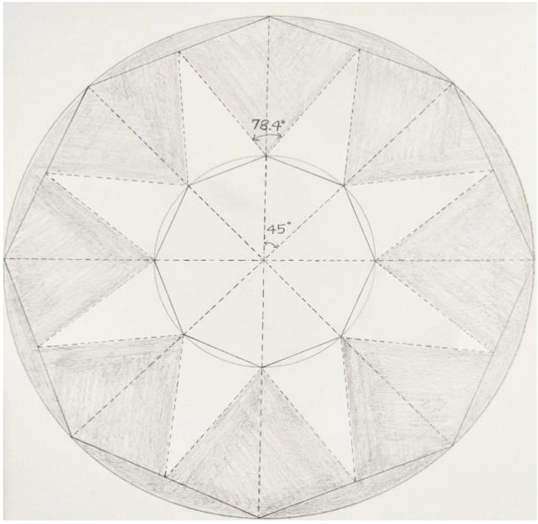


Figure-53: Undesired area (including V-parts) to be cut-removed from big circle is shaded

6) Cut & remove undesired area (as shaded in fig-53 above) from big circle (sheet) to get a blank of sheet (as shown in fig-54). Rotate all the isosceles triangular faces about the edges of small regular octagon (base) (as shown in fig-55)

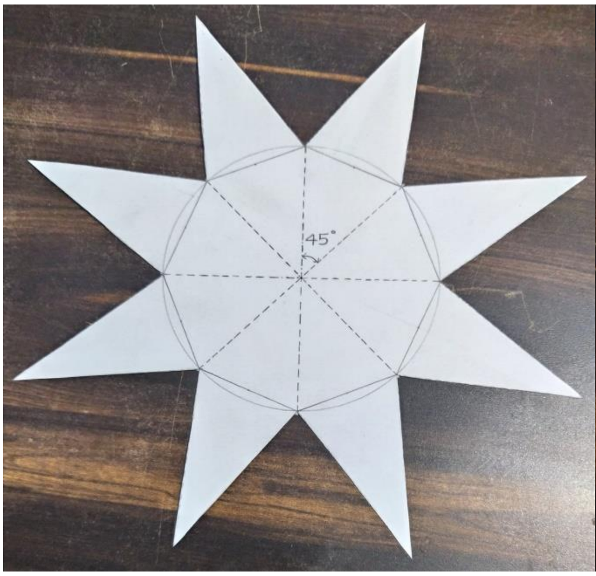


Figure-54: Blank of paper after cutting & removing undesired area (including V-parts) from big circle

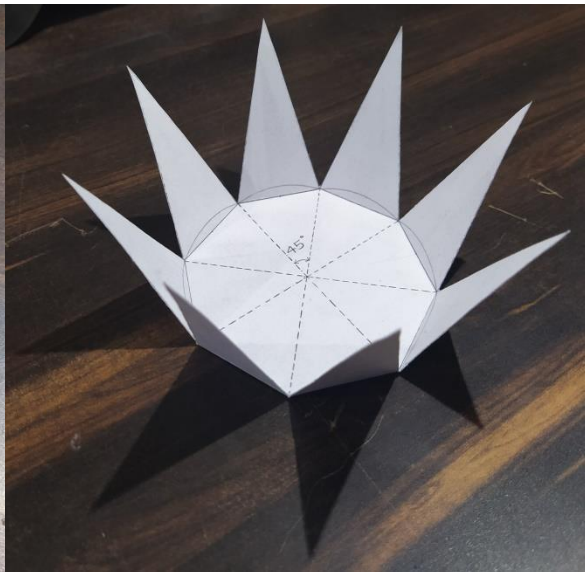


Figure-55: Rotating isosceles triangular faces about the edges of regular octagon (base) which become lateral faces after meeting at the edges

7.) Rotate the isosceles triangular faces about the edges of regular octagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired right pyramid with regular octagonal base (as shown in fig-56)

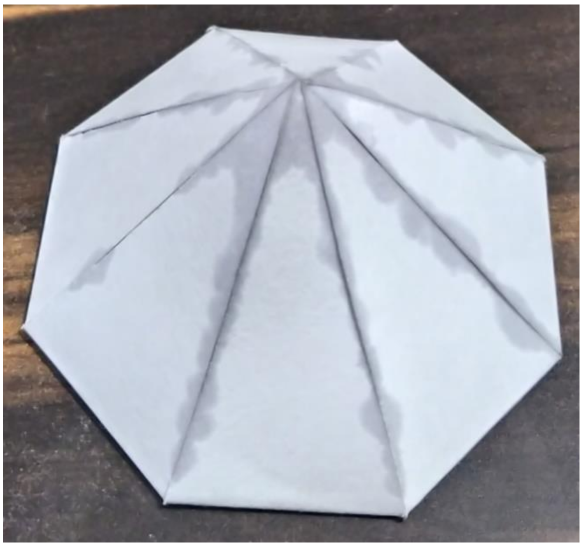


Figure-56: Desired right pyramid after butt-joining the mating lateral edges using suitable adhesive

The desired right pyramid with regular octagonal base has some important parameters which are analytically calculated by using generalized formula (as derived above) as follows

1. Length (L) of each lateral edge of right pyramid, is given as

$$L = \sqrt{l^2 + \left(\frac{a}{2}\right)^2} = \sqrt{(5)^2 + \left(\frac{3}{2}\right)^2} = 5.22\ cm$$

2. Normal height (h) of right pyramid, is given as

$$h = \sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}} = \sqrt{(5)^2 - \frac{(3)^2}{4}\cot^2\frac{\pi}{8}} = 3.448\ cm$$

3. Angle of inclination (θ) of each isosceles triangular lateral face with the plane of base ,is given as

$$\theta = \pi - \cos^{-1}\left(\frac{a}{2l}\cot\frac{\pi}{n}\right) = \pi - \cos^{-1}\left(\frac{3}{2\times 5}\cot\frac{\pi}{8}\right) = 136.41^o$$

4. Dihedral angle (θ_d) between any two consecutive lateral faces of right pyramid, is given as

$$\theta_d = 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{n} - \frac{a^2}{4l^2}\cos^2\frac{\pi}{n}}\right) = 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{8} - \frac{3^2}{4(5)^2}\cos^2\frac{\pi}{8}}\right) = 149.4^o$$

5. Total surface area (A_s) of right pyramid (base + lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + n\left(\frac{1}{2}al\right) = \frac{1}{4}(8)\,(3)^2\cot\frac{\pi}{8} + 8\left(\frac{1}{2}(3)(5)\right) = 103.456\ cm^2$$

6. Volume (V) of right pyramid, is given as

$$V = \frac{1}{12}na^2\cot\frac{\pi}{n}\sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}} = \frac{1}{12}(8)(3)^2\cot\frac{\pi}{8}\sqrt{5^2 - \frac{3^2}{4}\cot^2\frac{\pi}{8}} = 49.94\ cm^3$$

All above values of important parameters of right pyramid with regular octagonal base can be verified by accurate and precise measurements.

Polyhedron with two regular n-gonal and 2n trapezoidal faces:. So far, we have analysed in details the pyramidal flat container with regular n-gonal base & trapezoidal lateral faces. If we butt-join two identical pyramidal flat containers (having regular n-gonal base) at the edges of their regular n-gonal open ends then we get a closed surface i.e. a polyhedron with two regular n-gonal and 2n trapezoidal faces (A polyhedron is shown in fig-57 below). All the important parameters, like edge length, dihedral angle, surface area and the volume of polyhedron with two regular n-gonal and 2n trapezoidal faces, can be determined by using generalized formula for pyramidal flat container with regular n-gonal base as derived earlier.

Thus we can make a desired polyhedron with 2 regular n-gonal and 2n trapezoidal faces by joining two identical pyramidal flat containers at the edges of their open ends i.e. one pyramidal flat container is inverted (upside down) over the other identical one (upright) such that the edges of their regular n-gonal open-ends coincide then the mating edges are butt-joined to get e desired polyhedron (as shown in fig-57)

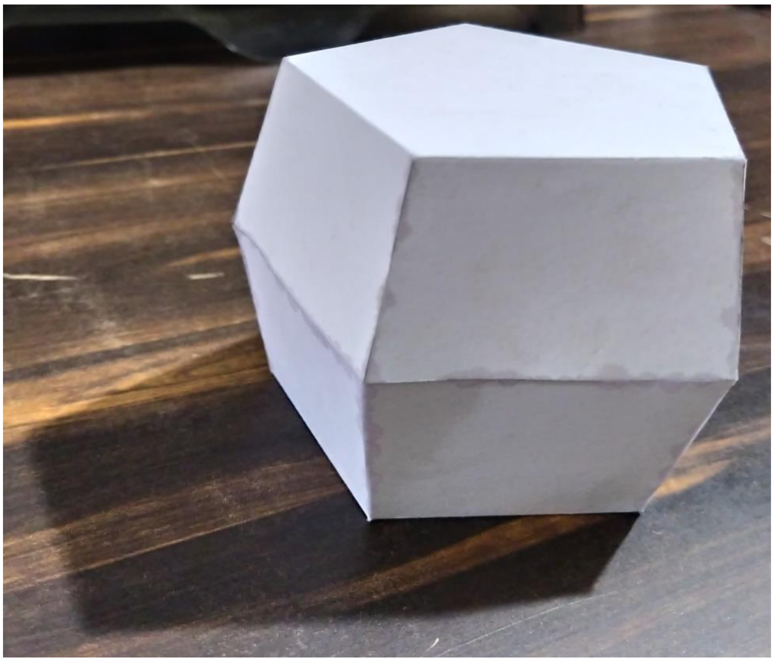


Figure-57: A paper model of polyhedron with two regular pentagonal and ten trapezoidal faces is made by butt-joining two identical pyramidal flat containers, with regular pentagonal base, at the edges of their regular pentagonal open-ends

Note: Above articles had been *derived & illustrated* by **Mr H.C. Rajpoot** (**M Tech, Production Engineering**)

Indian Institute of Technology Delhi *10 September, 2019*

Email:hcrajpoot.iitd@gmail.com

Author's Home Page: https://notionpress.com/author/HarishChandraRajpoot